Tug-the-hook symmetry for quantum 6j-symbols

Alexey Sleptsov

ITEP (Kurchatov institute), IITP & MIPT, Moscow

in collaboration with E.Lanina, V.Mishnyakov, N.Tselousov

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Quantum algebra $U_q(sl_N)$

Let $(c_{ij})_{1\le i,j\le N-1}$ be the Cartan matrix. The quantized universal enveloping algebra $U_q(sl_N)$ is defined by the generators $E_i, F_i, K_i = q^{H_i}, K_i^{-1}$ and the relations:

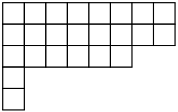
1. $K_iK_j = K_jK_i = 1, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$
2. $K_iE_jK_i^{-1} = q^{c_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-c_{ij}}F_j$
3. $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}$
4. $\displaystyle\sum_{s=0}^{1-c_{ij}}(-1)^s \left[\begin{array}{c} 1-c_{ij} \\ s \end{array}\right] E_i^{1-c_{ij}-s}E_jE_i^s = 0, \quad i \ne j$

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Representations and Multiplicity space

Finite-dimensional representations of $U_q(sl_N)$ enumerated by Young diagrams.

$$\lambda = [r_1 \geq r_2 \geq \ldots \geq r_{N-1}] =$$



Let us consider the tensor product of 2 f.-d. irreps $V_\mu \otimes V_\nu$ and decompose it into irreps:

$$V_\mu \otimes V_\nu = \bigoplus_\rho M^\rho_{\mu\nu} \otimes V_\rho \qquad (1)$$

$$\text{for } U_q(sl_2) \quad V_{j_1} \otimes V_{j_2} = V_{|j_1 - j_2|} \oplus \ldots \oplus V_{j_1 + j_2} \qquad (2)$$

Here $M^\rho_{\mu\nu}$ is the multiplicity space, i.e. the vector space of highest weight ρ in the product, whose dimension $m = \dim(M^\rho_{\mu\nu})$ is equal to the number of V_ρ in the decomposition. If $m = 1$ is one-dimensional, the representation is called **multiplicity-free**.

Quantum dimension

Let ρ be a half-sum of all positive roots of sl_N, then there exists an element $K_{2\rho} \in U_q(sl_N)$ defined as

$$K_{2\rho} = K_1^{n_1} K_2^{n_2} \dots K_{N-1}^{n_{N-1}}, \tag{3}$$

$$2\rho = \sum_{i=1}^{N-1} n_i \alpha_i, \;\; n_i \in \mathbb{N}_0, \tag{4}$$

where α_i are simple roots. The number

$$D_\mu := {}_q Tr_{V_\mu}(1) \equiv Tr_{V_\mu}(K_{2\rho}) \tag{5}$$

is called the quantum dimension of V_μ.

Two bases

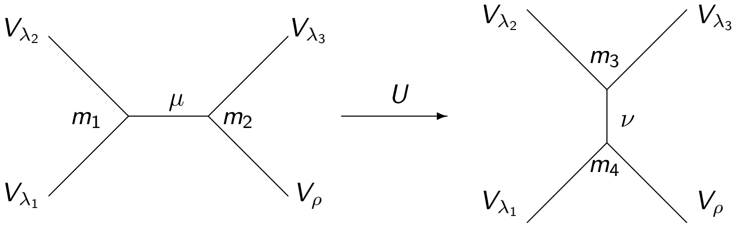
Let us consider three representations $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$.

Associativity of tensor product implies that $(V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3}$ is isomorphic to $V_{\lambda_1} \otimes (V_{\lambda_2} \otimes V_{\lambda_3})$:

$$(V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3} = \left(\bigoplus_{\mu} M_{\mu}^{\lambda_1 \lambda_2} \otimes V_{\mu} \right) \otimes V_{\lambda_3} = \bigoplus_{\mu,\rho} M_{\mu}^{\lambda_1 \lambda_2} \otimes M_{\rho}^{\mu \lambda_3} \otimes V_{\rho} \quad (6)$$

$$V_{\lambda_1} \otimes (V_{\lambda_2} \otimes V_{\lambda_3}) = V_{\lambda_1} \otimes \left(\bigoplus_{\nu} M_{\nu}^{\lambda_2 \lambda_3} \otimes V_{\nu} \right) = \bigoplus_{\rho,\nu} M_{\rho}^{\lambda_1 \nu} \otimes M_{\nu}^{\lambda_2 \lambda_3} \otimes V_{\rho} \quad (7)$$

Graphically one can depict these bases as follows



6j-symbols

The rotation matrix of one basis (6) into another (7) is called a matrix of 6j-symbols or Racah-Wigner matrix:

$$U \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \rho \end{bmatrix}_{m_1 m_2}^{m_3 m_4} : \bigoplus_{\mu} M_{\mu}^{\lambda_1 \lambda_2} \otimes M_{\rho}^{\mu \lambda_3} \longrightarrow \bigoplus_{\nu} M_{\rho}^{\lambda_1 \nu} \otimes M_{\nu}^{\lambda_2 \lambda_3} \tag{8}$$

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \mu \\ \lambda_3 & \rho & \nu \end{Bmatrix}_{m_1 m_2}^{m_3 m_4} = \frac{1}{\sqrt{D_{\mu} \cdot D_{\nu}}} U_{\mu,\nu} \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \rho \end{bmatrix}_{m_1 m_2}^{m_3 m_4} \tag{9}$$

6j symbols first appeared in the work of E. Wigner in 1940, where they were used as a tool for studying irreducible representations $SO(3)$ and $SU(2)$. In 1942, while studying atomic spectra, G. Racah introduced the concept of recoupling coefficients to describe angular momenta in quantum mechanics.

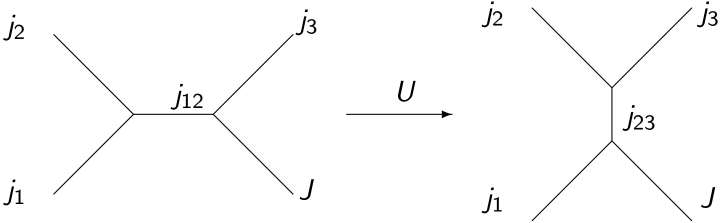
6j-symbols: Applications

6j-symbols are used in the generalized theory of angular momentum to describe complex systems: atoms, nuclei, molecules, hadrons. The non-complete list of applications:

1. nuclear physics (see Landau-Lifshitz. Quantum mechanics, V.3, §108)
2. QCD (e.g., Landau-Pomeranchuk-Migdal effect)
3. condensed matter (e.g., ultracold alkaline-earth atoms)
4. conformal field theories (fusion matrix)
5. 3d quantum gravity (Ponzano-Regge model)
6. integrable systems
7. knot theory (Reshetikhin-Turaev invariants)
8. invariant of 3-manifolds (Turaev-Viro invariants)
9. topological quantum computer
10. special functions (e.g., orthogonal polynomials)

Addition of angular momenta

The 6j-symbols arise naturally in the problem concerning the addition of 3 angular momenta. The state of the system depends on coupling scheme:

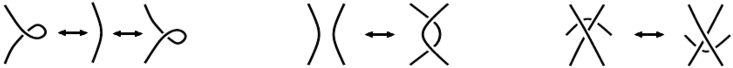


Wave functions of the coupling schemes are related as follows:

$$\psi_{j_{23},J} = \sum_{j_{12}} U_{j_{12},j_{23}} \cdot \psi_{j_{12},J} \tag{10}$$

Example of quantum knot invariant I

$\mathcal{K}_1 = \mathcal{K}_2 \;\Leftrightarrow\;$ Reidemeister moves:



I II III

Let's associate vector space with every strand $|_i \mapsto V_i$. If we define invertible linear operators by

$$\mathcal{R}_i = 1_{V_1} \otimes 1_{V_2} \otimes \ldots \otimes P\check{\mathcal{R}}_{i,i+1} \otimes \ldots \otimes 1_{V_m} \;\in \mathrm{End}(V_1 \otimes \ldots, \otimes V_m), \qquad (11)$$

where $P(x \otimes y) = y \otimes x$ and $\check{\mathcal{R}}$ acts on two $U_q(sl_N)$-modules V_i and V_{i+1}

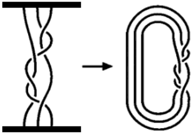
$$\check{\mathcal{R}}_{V,W} = q^{\sum_{i,j} C^{-1}_{ij} H_i \otimes H_j} \prod_{\text{posit. roots } \beta} \exp_q[(1 - q^{-1}) E_\beta \otimes F_\beta] \qquad (12)$$

then it is well known that $\mathcal{R}_1, \ldots, \mathcal{R}_{m-1}$ define a representation of the Artin's braid group B_m on m strands:

$$\begin{aligned} \pi : B_m &\to \mathrm{End}(V_1 \otimes \ldots, \otimes V_m) \\ \pi(\sigma_i) &= \mathcal{R}_i, \end{aligned} \qquad (13)$$

where $\sigma_1, \ldots, \sigma_{m-1}$ are generators of the braid group B_m.

Example of quantum knot invariant II



Any knot can be given as the closure of the corresponding braid. Operators $\mathcal{R}_1, \ldots, \mathcal{R}_{m-1}$ satisfies 2nd and 3 Reidemeister moves. In order to satisfy 1st R-move one should to consider quantum trace:

$$\boxed{H_\lambda^{\mathcal{K}}(q, A = q^N) = {}_q\mathrm{Tr}_{V_1 \otimes \cdots \otimes V_m}\left(\mathcal{R}_1^{a_1} \mathcal{R}_2^{a_2} \ldots\right),} \qquad (14)$$

where $V_1 = \ldots = V_m = V_\lambda$, because we consider a knot, which have only 1 component.

This quantum knot invariant in the case of $U_q(sl_N)$ is called colored HOMFLY-PT polynomial.

This invariant (14) is a vacuum expectation value $\langle W_R(\mathcal{K})\rangle$ for the Wilson loop correlators in the Chern-Simons theory. It is very interesting, because the Chern-Simons theory is defined in terms of classical Lie algbera, but non-perturbative answer is given in terms of quantum algebra.

Example of quantum knot invariant III

How to calculate this quantity $H_\lambda^{\mathcal{K}}(q, A = q^N) = {}_q\mathrm{Tr}_{V_1\otimes\cdots\otimes V_m}\left(\mathcal{R}_1^{a_1}\mathcal{R}_2^{a_2}\dots\right)$?

• Consider the case with 2 strands, where we have only $\mathcal{R}_1 = P\check{\mathcal{R}}_{V_1,V_2}$. Its eigenvalues are known [N.Reshetiknin'1987]:

$$e_k(\mathcal{R}_{V_1,V_2}) = \pm q^{C_2(Q_k)-C_2(V_1)-C_2(V_2)}, \quad \text{where } C_2 \text{ is a quadratic} \tag{15}$$
$$\text{Casimir operator } C_2(V_\lambda) = \sum_{(i,j)\in\lambda}(i-j), \quad V_1\otimes V_2 = \bigoplus_k Q_k.$$

Example of quantum knot invariant III

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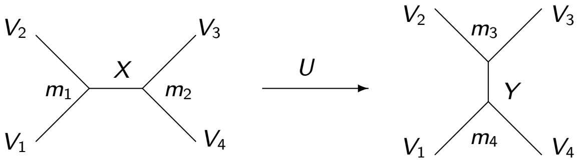
• For 3 strands we have 2 R-matrices: $\mathcal{R}_1 = \check{\mathcal{R}}\otimes 1$ and $\mathcal{R}_2 = 1\otimes\check{\mathcal{R}}$.
Diagonalization: basis $(V_1\otimes V_2)\otimes V_3$ for $\mathcal{R}_1$ and $V_1\otimes(V_2\otimes V_3)$ for $\mathcal{R}_2$.



With the help of Racah-Wigner matrix we have

$$\mathcal{R}_2 = U^\dagger\begin{bmatrix} V_1 & V_3 \\ V_2 & V_4\end{bmatrix}\cdot\mathrm{diag}\left(\mathcal{R}_2\right)\cdot U\begin{bmatrix} V_1 & V_2 \\ V_3 & V_4\end{bmatrix}.$$

Symmetries of 6j-symbols: orthogonality

Values and various properties of quantum 6j-symbols are well-known for $U_q(sl_2)$ [*A.N.Kirillov and N.Y.Reshetikhin, 1989*], but much less is known for $N > 2$. Nevertheless, some properties and relations for them are known for an arbitrary N. Let us briefly discuss them [*C.R.Lienert and P.H.Butler, 1992*].

- Racach matrices are orthogonal:

$$\sum_{\lambda_{12}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda'_{23} \end{Bmatrix} D_{\lambda_{12}} = \frac{\delta_{\lambda_{23},\lambda'_{23}}}{\sqrt{D_{\lambda_{23}} \cdot D_{\lambda'_{23}}}} \tag{16}$$

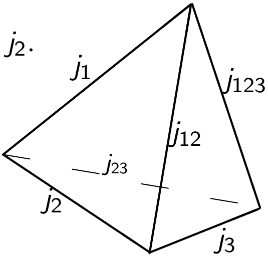
Symmetries of 6j-symbols: tetrahedron I

Representations of $U_q(sl_2)$ enumerated by $[j] = \square\square\square\square\square\square\square$.

Tensor product rule is given by :

$[j_1] \otimes [j_2] = [j_{12}], \quad j_{12} = |j_1 - j_2|, \ldots, j_1 + j_2.$

$$\left\{ \begin{array}{ccc} [j_1] & [j_2] & [j_{12}] \\ [j_3] & [j_{123}] & [j_{23}] \end{array} \right\} =$$



Tetrahedral symmetry group S_4 contains $4! = 24$ elements.

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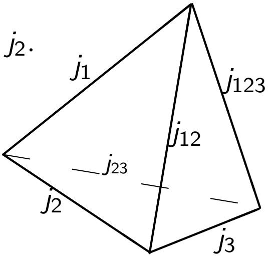
$$\left\{ \begin{matrix} [j_1] & [j_2] & [j_{12}] \\ [j_3] & [j_{123}] & [j_{23}] \end{matrix} \right\} =$$



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Additional **Regge symmetries** [*T.Regge,1959*]:

$$\left\{ \begin{matrix} [j_1] & [j_2] & [j_{12}] \\ [j_3] & [j_4] & [j_{23}] \end{matrix} \right\} = \left\{ \begin{matrix} [p-j_1] & [p-j_2] & [j_{12}] \\ [p-j_3] & [p-j_4] & [j_{23}] \end{matrix} \right\}, \qquad (17)$$

where $p = \frac{1}{2}(j_1+j_2+j_3+j_4)$. In total, for $N = 2$ we have 144 symmetries, full symmetry group $S_4 \times S_3$.

Symmetries of 6j-symbols: tetrahedron II

Tetrahedral symmetries are the symmetries between 6-j symbols that are generated by permutations of rows and columns

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} = \begin{Bmatrix} \overline{\lambda_3} & \overline{\lambda_2} & \overline{\lambda_{23}} \\ \overline{\lambda_1} & \overline{\lambda_{123}} & \overline{\lambda_{12}} \end{Bmatrix} = \tag{18}$$

$$= \begin{Bmatrix} \lambda_3 & \overline{\lambda_{123}} & \overline{\lambda_{12}} \\ \lambda_1 & \overline{\lambda_2} & \overline{\lambda_{23}} \end{Bmatrix} = \begin{Bmatrix} \lambda_2 & \overline{\lambda_{12}} & \overline{\lambda_1} \\ \lambda_{123} & \lambda_{23} & \lambda_3 \end{Bmatrix} = \tag{19}$$

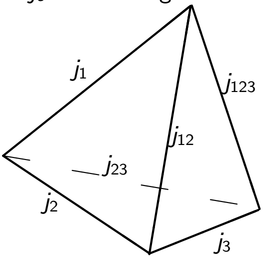
$$= \begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_{12} \\ \overline{\lambda_{123}} & \overline{\lambda_3} & \overline{\lambda_{23}} \end{Bmatrix}, \tag{20}$$

where $\overline{\lambda}$ denotes conjugate representation: $\lambda \otimes \overline{\lambda} \ni \varnothing$

Turaev-Viro invariants of 3-manifolds

Let us fix a triangulation $\triangle$ of a compact 3-manifold $\mathcal{M}$. For simplicity we assume that $\mathcal{M}$ is closed $\partial\mathcal{M} = \varnothing$. Let us consider irreps of $U_q(sl_2)$ and put $q = \exp\left(\frac{2\pi i}{k}\right)$, $k > 2$. We associate representation j_e to each edge e as follows:

$$\left\{ \begin{array}{ccc} [j_1] & [j_2] & [j_{12}] \\ [j_3] & [j_{123}] & [j_{23}] \end{array} \right\} \quad =$$



Then we define the following partition function:

$$TV(q,\triangle) := \left(-\frac{\left(q-q^{-1}\right)^2}{2k}\right)^v \sum_{j=1}^{k} \prod_{e\in\triangle} D_{j_e} \prod_{t\in\triangle} \left\{ \begin{array}{ccc} [j_1] & [j_2] & [j_{12}] \\ [j_3] & [j_{123}] & [j_{23}] \end{array} \right\} \qquad (21)$$

According to the theorem of V.G.Turaev and O.Y.Viro [1990] the quantity $TV(q,\triangle)$ is independent of the triangulation $\triangle$, but depends only on the topology of $\mathcal{M}$.

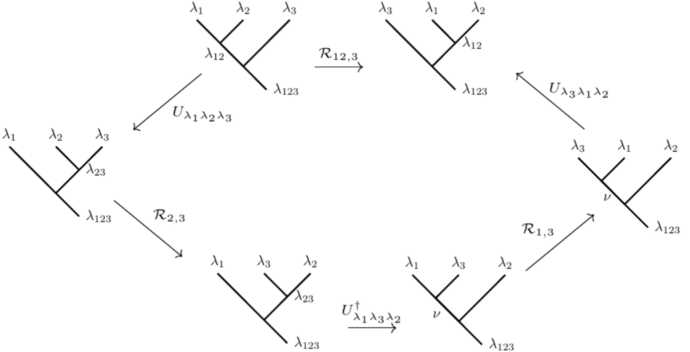
Symmetries of 6j-symbols: Racah identity

The Racah back-coupling rule is a general property of 6-j symbols:

$$q^{C_2(\lambda_1)+C_2(\lambda_3)+C_2(\lambda_{12})+C_2(\lambda_{23})-C_2(\lambda_2)-C_2(\lambda_{123})} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} =$$

$$= \sum_{\nu} \pm D_{\nu} q^{C_2(\nu)} \begin{Bmatrix} \lambda_{23} & \nu & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_1 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \overline{\lambda_{23}} & \nu & \overline{\lambda_3} \end{Bmatrix}$$

This property follows from the hexagon axiom of R-matrix $\hat{\mathcal{R}}_{12,3} = \hat{\mathcal{R}}_{1,2}\hat{\mathcal{R}}_{2,3}$:



Symmetries of 6j-symbols: pentagon I

There are 5 possibilities to decompose the tensor product of $T_{\lambda_1} \otimes T_{\lambda_2} \otimes T_{\lambda_3} \otimes T_{\lambda_4}$ into irreducible representations:

$$((T_{\lambda_1} \otimes T_{\lambda_2}) \otimes T_{\lambda_3}) \otimes T_{\lambda_4} \tag{22}$$

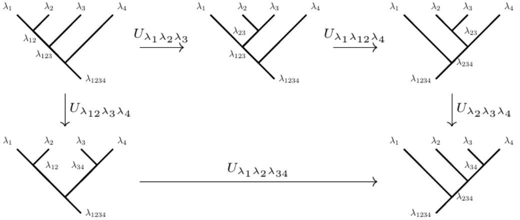
$$(T_{\lambda_1} \otimes (T_{\lambda_2} \otimes T_{\lambda_3})) \otimes T_{\lambda_4} \tag{23}$$

$$(T_{\lambda_1} \otimes T_{\lambda_2}) \otimes (T_{\lambda_3} \otimes T_{\lambda_4}) \tag{24}$$

$$T_{\lambda_1} \otimes ((T_{\lambda_2} \otimes T_{\lambda_3}) \otimes T_{\lambda_4}) \tag{25}$$

$$T_{\lambda_1} \otimes (T_{\lambda_2} \otimes (T_{\lambda_3} \otimes T_{\lambda_4})) \tag{26}$$

We can go from (22) to (26) by (22) $\to$ (23) $\to$ (25) $\to$ (26) and also by the chain (22) $\to$ (24) $\to$ (26) using 6j-symbols at each step.



Symmetries of 6j-symbols: pentagon II

By this way we get Biedenharn-Elliott identity ('1953), or pentagon identity:

$$\sum_{\lambda_{23}} D_{\lambda_{23}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \lambda_{123} \\ \lambda_4 & \lambda_{1234} & \lambda_{234} \end{Bmatrix} \begin{Bmatrix} \lambda_2 & \lambda_3 & \lambda_{23} \\ \lambda_4 & \lambda_{234} & \lambda_{34} \end{Bmatrix} =$$

$$= \begin{Bmatrix} \lambda_{12} & \lambda_3 & \lambda_{123} \\ \lambda_4 & \lambda_{1234} & \lambda_{34} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_{34} & \lambda_{1234} & \lambda_{234} \end{Bmatrix}.$$

Theorem (Butler'1981)

Non-primitive 6j-symbols can always be converted to primitive ones.

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \varnothing \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix}, \begin{Bmatrix} \lambda_1 & \lambda_2 & \square \text{ or } \overline{\square} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix}$$

Recipe. Let us calculate $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_{34} & \lambda_{1234} & \lambda_{234} \end{Bmatrix}$. Let λ_{12} be a representation with the smallest number of boxes. Put $\lambda_3 = \overline{\square}$ and take λ_{123} one box fewer than λ_{12} (fusion rules are ok).

Known results

- $U_q(sl_2)$, *A.N.Kirillov and N.Y.Reshetikhin, 1989*

$$\left\{ \begin{matrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{matrix} \right\} = \sqrt{[2i+1][2j+1]} \cdot (-1)^{\sum_{m=1}^{4} r_m} \cdot \theta(r_1, r_2, i)\, \theta(r_3, r_4, i)\, \theta(r_4, r_1, j) \theta(r_2, r_3, j) \times$$

$$\times \sum_{k \geq 0} \frac{(-1)^k [k+1]! \cdot [k-r_1-r_2-i]!^{-1} [k-r_3-r_4-i]!^{-1} [k-r_1-r_4-j]!^{-1} [k-r_2-r_3-j]!^{-1}}{[r_1+r_2+r_3+r_4-k]! [r_1+r_3+i+j-k]! [r_2+r_4+i+j-k]!},$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \;\; \theta(a, b, c) = \sqrt{\frac{[a-b+c]![b-a+c]![a+b-c]!}{[a+b+c+1]!}}$$

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$$\left\{\begin{array}{ccc} r_1 & r_2 & i \\ r_3 & r_4 & j \end{array}\right\} = \sqrt{[2i+1][2j+1]}\cdot(-1)^{\sum_{m=1}^{4} r_m}\cdot\theta(r_1,r_2,i)\,\theta(r_3,r_4,i)\,\theta(r_4,r_1,j)\theta(r_2,r_3,j)\times$$

$$\times\sum_{k\geq 0}\frac{(-1)^k[k+1]!\cdot[k-r_1-r_2-i]!^{-1}[k-r_3-r_4-i]!^{-1}[k-r_1-r_4-j]!^{-1}[k-r_2-r_3-j]!^{-1}}{[r_1+r_2+r_3+r_4-k]![r_1+r_3+i+j-k]![r_2+r_4+i+j-k]!},$$

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}},\ \ \theta(a,b,c)=\sqrt{\frac{[a-b+c]![b-a+c]![a+b-c]!}{[a+b+c+1]!}}$$

- $U_q(sl_N)$, *S. Alisauskas, 1995*, $\left\{\begin{array}{ccc} [r_1] & \lambda_2 & \lambda_{12} \\ [r_3] & \lambda_{123} & \lambda_{23} \end{array}\right\}^{11}_{11}$

Solution of the pentagon identity in $U_q(sl_2)$

K.S.Rao, T.S.Santhanam, R.A.Gustafson, 1987

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$\lambda_1 = [r_1], \quad \lambda_2 = [r_2], \quad \lambda_3 = [r_3], \quad \lambda_{12} = [r_{12}], \quad \lambda_{23} = [r_{23}]$

$\lambda_4 = [2], \quad \lambda_{34} = [r_3] \;\in\; \lambda_3 \otimes \lambda_4 = \{[\mathbf{r}'_3 - \mathbf{2}], [\mathbf{r}'_3], [\mathbf{r}'_3 + \mathbf{2}]\}$

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$r_{23} \in [2] \otimes [n] = [n-2] \oplus [n] \oplus [n+2], \quad x := r_{12}$

$$\sum_{i=-1}^{1} c_i \begin{Bmatrix} r_1 & r_2 & \mathbf{x} \\ r_3 & R & \mathbf{n} + 2i \end{Bmatrix} = \begin{Bmatrix} \mathbf{x} & r_3 & R \\ 2 & R & r_3 \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & \mathbf{x} \\ r_3 & R & \mathbf{n} \end{Bmatrix} \tag{30}$$

q-Hypergeometric series

The q-hypergeometric series is defined as:

$$ {}_r\phi_p \left(\begin{matrix} a_1 & \dots & a_j \\ b_1 & \dots & b_k \end{matrix} ; q, z \right) \equiv \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_p; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+p-r} \frac{z^n}{(q,q)} \quad (31) $$

where $(a, q)_n \equiv \prod_{k=0}^{n-1}(1 - aq^k)$ is a q-Pochhammer symbol. In the case $r = p + 1$ a more convenient form is:

$$ {}_{p+1}\Phi_p \left(\begin{matrix} a_1, \dots, a_p, a_{p+1} \\ b_1, \dots, b_p \end{matrix} ; q, z \right) \equiv {}_{p+1}\phi_p \left(\begin{matrix} q^{a_1}, \dots, q^{a_p}, q^{a_{p+1}} \\ q^{b_1}, \dots, q^{b_p} \end{matrix} ; q, z \right). \quad (32) $$

because it may be reformulated in terms of q-factorials:

$$ {}_{p+1}\Phi_p \left(\begin{matrix} a_1+1, \dots, a_p+1, a_{p+1}+1 \\ b_1 + 1, \dots, b_p + 1 \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{[a_1+n]!}{[a_1]!} \dots \frac{[a_{p+1}+n]!}{[a_{p+1}]!} \frac{[b_1]!}{[b_1+n]!} \dots \frac{[b_p]!}{[b_p+n]!} \frac{z^n}{[n]!}. $$

6j-symbol via q-hypergeometric series

With the help of Sears' transformation

$$ {}_4\Phi_3\left(\begin{matrix} x,y,z,n \\ u,v,w \end{matrix};q,q\right) = \frac{[v-z-n-1]![u-z-n-1]![v-1]![u-1]!}{[v-z-1]![v-n-1]![u-z-1]![u-n-1]!}\, {}_4\Phi_3\left(\begin{matrix} w-x,w-y,z,n \\ 1-u+z+n,1-v+z+n,w \end{matrix};q,q\right) \tag{33} $$

one can transform Kirillov-Reshetikhin's answer into the following

$$ \begin{Bmatrix} r_1 & r_2 & r_{12} \\ r_3 & R & r_{23} \end{Bmatrix} = K \cdot {}_4\Phi_3\left(\begin{matrix} a_1,a_2,a_3,a_4 \\ b_1,b_2,b_3 \end{matrix};q,q\right), \tag{34} $$

$$ 2a_i = \begin{pmatrix} -r_1-r_2+r_{12} \\ -r_1-r_2-r_{12}-2 \\ -r_1-R+r_{23} \\ -r_1-R-r_{23}-2 \end{pmatrix}, \quad 2b_i = \begin{pmatrix} -r_1-r_2-r_3-R-2 \\ -2r_1 \\ -2r_1 \end{pmatrix}, \tag{35} $$

$$ K = \sqrt{\frac{\left[\frac{r_1-R+r_{23}}{2}\right]!\left[\frac{r_{23}-r_3+r_2}{2}\right]!}{\left[\frac{-r_1+R+r_{23}}{2}\right]!\left[\frac{R+r_1+2+r_{23}}{2}\right]!}}\sqrt{\frac{1}{\left[\frac{r_3-r_2+r_{23}}{2}\right]!\left[\frac{r_2+r_{23}+2+r_3}{2}\right]!\left[\frac{R+r_1-r_{23}}{2}\right]!\left[\frac{r_3+r_2-r_{23}}{2}\right]!}} $$

q-Racah polynomial

$$\mathfrak{R}_n(z(x);a,b,c,d|q) = {}_4\Phi_3\left(\begin{matrix}-n, n{+}a{+}b{+}1, -x, x{+}c{+}d{+}1 \\ a+1, b+d+1, c+1\end{matrix}; q, q\right)$$

where $n = 0, 1, \ldots, L$ is the degree of the polynomial in variable
$z(x) := [x][x{+}c{+}d{+}1]$
Three-term recurrence relation:

$$[x][x{+}c{+}d{+}1]\ \mathfrak{R}_n\big(z(x)\big) = A_n \cdot \mathfrak{R}_{n+1}\big(z(x)\big) - (A_n{+}C_n)\cdot \mathfrak{R}_n\big(z(x)\big) + C_n \cdot \mathfrak{R}_{n-1}\big(z(x)\big) \tag{36}$$

with coefficients specified for q-Racah polynomial:

$$A_n = \frac{[n+a+1][n+a+b+1][n+b+d+1][n+c+1]}{[2n+a+b+1][2n+a+b+2]}$$

$$C_n = \frac{[n][n+a+b-c][n+a-d][n+b]}{[2n+a+b][2n+a+b+1]}$$

$$\sum_x w(x)\ \mathfrak{R}_n(z(x))\ \mathfrak{R}_m(z(x)) = h_n d_{nm} \tag{37}$$

6j and q-Racah polynomial

$$\begin{Bmatrix} r_1 & r_2 & r_{12} \\ r_3 & R & r_{23} \end{Bmatrix} = \frac{1}{K} \cdot \mathfrak{R}_n(\nu(x); \alpha, \beta, \gamma, \delta | q), \quad \text{where} \tag{38}$$

$$\alpha = -r_3 - 1, \;\beta = -r_2 - 1, \;\delta = \frac{r_1 - R - r_3 + r_2}{2}, \;\gamma = -\frac{R + r_1 + r_3 + r_2}{2} - 2,$$
$$n = \frac{r_3 + r_2 - r_{23}}{2}, \;x = \frac{r_3 + R - r_{12}}{2}$$

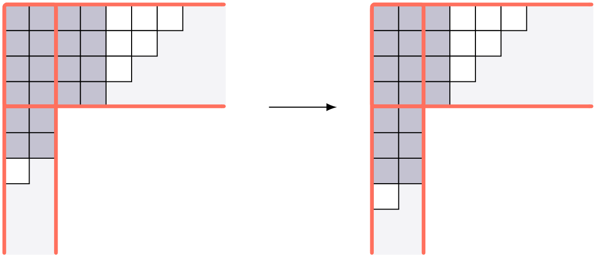
Orthogonal relation for q-Racah polynomials

$$\sum_x P_n(\nu(x)) P_n(\nu(x)) \cdot \frac{D(\nu)D(\mu)}{K^2} = 1 \tag{39}$$

comes from orthogonal relation for 6j-symbols up to normalization K^2:

$$\sum_{r_{12}} \begin{Bmatrix} r_1 & r_2 & r_{12} \\ r_3 & R & r_{23} \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & r_{12} \\ r_3 & R & r_{23} \end{Bmatrix} D(\nu)D(\mu) = 1 \tag{40}$$

Tug-the-hook symmetry I



A Young diagram is placed inside an appropriate $(K+M|M)$ fat hook. Parametrize the first K rows by their length R_i, $i = 1, \dots, K$, the rest rows are parametrized by shifted Frobenius variables

$$\alpha_i = R_i - (i-K) + 1, \;\; \beta_i = R'_{i-K} - i + 1, \;\; i = K+1, \dots, K+M \tag{41}$$

The tug-the-hook transformation pulls the Young diagram inside the fat hook:

$$\mathbf{T}_\epsilon^{(K+M|M)} : R_i \longrightarrow R_i - \epsilon \,, \quad \alpha_i \longrightarrow \alpha_i - \epsilon \,, \quad \beta_i \longrightarrow \beta_i + \epsilon \,, \tag{42}$$

where ϵ is the corresponding shift of the diagram.

Tug-the-hook symmetry II

Conjecture (Lanina,Sleptsov'2022)

Given arbitrary irreducible finite-dimensional representations

$$V_{\lambda_1},\ \ V_{\lambda_2},\ \ V_{\lambda_3},$$

$$V_{\lambda_{12}} \in V_{\lambda_1} \otimes V_{\lambda_2}, \qquad V_{\lambda_{23}} \in V_{\lambda_2} \otimes V_{\lambda_3},$$

$$V_{\lambda_{123}} \in V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$$

of $U_q(sl_N)$ for generic q, Racah coefficients are invariant under the tug-the-hook transformation:

$$U \left[\begin{array}{cc|c} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{array} \right] = U \left[\begin{array}{cc|c} \mathbf{T}_{\epsilon_1}^{(K+M|M)}(\lambda_1) & \mathbf{T}_{\epsilon_2}^{(K+M|M)}(\lambda_2) & \mathbf{T}_{\epsilon_1+\epsilon_2}^{(K+M|M)}(\lambda_{12}) \\ \mathbf{T}_{\epsilon_3}^{(K+M|M)}(\lambda_3) & \mathbf{T}_{\epsilon_1+\epsilon_2+\epsilon_3}^{(K+M|M)}(\lambda_{123}) & \mathbf{T}_{\epsilon_2+\epsilon_3}^{(K+M|M)}(\lambda_{23}) \end{array} \right]$$

for K, M and integer shifts ϵ_1, ϵ_2, ϵ_3 for which the tug-the-hook transformation is defined.

Evidence for the tug-the-hook symmetry

There are three evidences, which support this conjecture:

1. eigenvalue conjecture, which has been proved for several specific cases:
   - for $U_q(\mathfrak{sl}_2)$ in [Alekseev,Morozov,Sleptsov'2021];
   - in multiplicity-free $U_q(\mathfrak{sl}_N)$ case for coinciding incoming representations for matrices up to size 5×5 [Tuba,Wenzl'2001; Itoyama et all'2013].
2. tug-the-hook symmetry for colored HOMFLY-PT polynomial:

$$\mathcal{H}_R^{\mathcal{K}}\left(q, A = q^N\right) = \mathcal{H}^{\mathcal{K}}_{\mathbf{T}_\epsilon^{(N+M|M)}(R)}\left(q, A = q^N\right) \tag{43}$$

3. highly non-trivial examples for 6j-symbols with multiplicities:

$$U\begin{bmatrix} [3,1] & [3,1] \\ [3,1] & [8,2,1,1] \end{bmatrix} = U\begin{bmatrix} [3,2] & [3,2] \\ [3,2] & [7,5,1,1,1] \end{bmatrix}. \tag{44}$$

Thank you for your attention!