Non-Abelian Poisson brackets

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Introduction

Let x_α, $\alpha = 1, \ldots, N$, be $m \times m$ matrices. How to define a Poisson brackets on the vector space of functions of the form $\operatorname{tr} P(x_1, ..., x_N)$, where P ia s (non-commutative) polynomial with coefficients over $\mathbb{C}$?

A naive version

Consider Nm^2-dimensional Poisson brackets defined on functions of entries $x^j_{i,\alpha}$ of matrices x_α.

Definition. A Poisson bracket defined on the matrix entries is called a *trace bracket* if

- the bracket is GL_m-invariant,
- for any two matrix polynomials $P_i(x_1, ..., x_N), \quad i = 1, 2$ with coefficients from $\mathbb{C}$ the bracket between their traces is equal to the trace of some matrix polynomial P_3 .

Remark. For any Hamiltonian of the form $H = \operatorname{tr} P$, where P is a matrix polynomial, and for any Poisson trace bracket, equations of motion can be written in the matrix form

$$\frac{dx_\alpha}{dt} = F_\alpha(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_N). \tag{1}$$

Theorem. i). Any constant trace Poisson bracket has the form

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = \delta^j_{i'}\delta^{j'}_i c_{\alpha\beta}; \tag{2}$$

ii). Any linear trace bracket is written as

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = b^\gamma_{\alpha,\beta} x^{j'}_{i,\gamma}\delta^j_{i'} - b^\gamma_{\beta,\alpha} x^j_{i',\gamma}\delta^{j'}_i; \tag{3}$$

iii). Any quadratic trace bracket is given by the formula

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta} x^{j'}_{i,\gamma} x^j_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta} x^k_{i,\gamma} x^{j'}_{k,\epsilon}\delta^j_{i'} - a^{\gamma\epsilon}_{\beta\alpha} x^k_{i',\gamma} x^j_{k,\epsilon}\delta^{j'}_i. \tag{4}$$

Furthermore,

1). The bracket (2) is a Poisson bracket iff

$$c_{\alpha\beta} = -c_{\beta\alpha}.$$

2). The formula (3) defines a Poisson bracket, iff

$$b^{\mu}_{\alpha\beta} b^{\sigma}_{\mu\gamma} = b^{\sigma}_{\alpha\mu} b^{\mu}_{\beta\gamma}. \tag{5}$$

3) The formula (4) defines a Poisson bracket iff the relations

$$r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha}, \tag{6}$$

$$r^{\lambda\sigma}_{\alpha\beta} r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau} r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha} r^{\lambda\mu}_{\sigma\beta} = 0, \tag{7}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\mu\sigma}_{\tau\alpha} a^{\nu\lambda}_{\sigma\beta}, \tag{8}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\sigma\tau} = a^{\mu\sigma}_{\alpha\beta} r^{\lambda\nu}_{\tau\sigma} + a^{\mu\nu}_{\alpha\sigma} r^{\sigma\lambda}_{\beta\tau}. \tag{9}$$

and

$$a^{\lambda\sigma}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\sigma\nu}_{\alpha\beta} r^{\lambda\mu}_{\sigma\tau} + a^{\mu\nu}_{\sigma\beta} r^{\sigma\lambda}_{\tau\alpha} \tag{10}$$

hold.

Remark. The formula (5) means that $b^{\sigma}_{\alpha\beta}$ are structural constants of an associative algebra $\mathcal{A}$. A direct verification shows that the bracket (3) is nothing but the Lie – Kirillov –Konstant bracket defined by the Lie algebra corresponding to the associative algebra $\mathrm{Mat}_m \otimes \mathcal{A}$.

Remark. The relations (6) and (7) mean that the tensor $\mathbf{r}$ satisfies the associative Yang–Baxter equation (or the Rota–Baxter equation).

An important subclass of the Poisson brackets (4) corresponds to the case of zero tensor $\mathbf{a}$.

Definition. An associative algebra $\mathcal{A}$ with multiplication $\circ$ is called *anti-Frobenius algebra* if it possesses a **non-degenerate antisymmetric** bilinear form (,) that satisfies the condition

$$(x,\, y \circ z) + (y,\, z \circ x) + (z,\, x \circ y) = 0 \qquad (11)$$

for any $x, y, z \in \mathcal{A}$. In other words, the form (,) defines a 1-cocycle on $\mathcal{A}$.

Theorem. There is a one-to-one correspondence between solutions of the system (6), (7) up to equivalence and exact representations of anti-Frobenius algebras up to isomorphism.

Construction to one side. Suppose we have an exact m-dimensional representation of an anti-Frobenius algebra $\mathcal{A}$. Let matrices y_γ with entries $y^i_{j,\gamma}$ form a basis in $\mathcal{A}$. Denote the matrix of the bilinear form on $\mathcal{A}$ by G. Let $g^{\alpha\beta}$ be entries of G^{-1}. It can be verified that the tensor

$$r^{ij}_{kl} = \sum_{\alpha,\beta=1}^{p} g^{\alpha\beta} y^i_{k,\alpha} y^j_{l,\beta}, \qquad i,j,k,l = 1,\ldots,m,$$

satisfies the relations (6), (7).

Open problem. Describe all anti-Frobenius algebras $\mathcal{A}$ of the form

$$\mathcal{A} = \mathcal{S} \oplus \mathcal{M},$$

where $\mathcal{S}$ is a semisimple associative algebra, and $\mathcal{M}$ is an $\mathcal{S}$–bimodule such that $\mathcal{M}^2 = \{0\}$.

Non-abelian Poisson brackets on free associative algebras

The Poisson brackets, which we considered above, were defined on functions of the entries of the matrices x_α, $\alpha = 1, \ldots, N$ and allowed a restriction on the vector space of traces of matrix polynomials.

Question. How to generalize these Poisson brackets to the case of free associative algebra?

A Poisson structure on a commutative associative algebra A is given by a Lie bracket

$$\{\cdot,\,\cdot\} : A \times A \mapsto A,$$

satisfying the the Leibniz rule

$$\{a,\, b\,c\} = \{a,\, b\}\, c + b\,\{a,\, c\}, \qquad a, b, c \in A.$$

A naive generalization of this definition to the case of a non-commutative associative algebra A is not informative due to the lack of examples of such brackets, other than the usual commutator.

We consider the version of the Hamiltonian formalism on the free associative algebra $\mathcal{A}$, proposed by M. Kontsevich. Our **non-abelian Poisson brackets** are defined *only* between traces of elements from $\mathcal{A}$.

The traces are regarded as elements of the quotient space $\mathcal{T} = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$.

Let $\mathcal{A}$ be the free associative algebra $\mathbb{C}[x_1, \ldots, x_N]$ with the product $\circ$. For any element $a \in \mathcal{A}$ we denote by L_a (respectively R_a) the operator of left (respectively, right) multiplication by a:

$$L_a(X) = a \circ X, \qquad R_a(X) = X \circ a, \qquad X \in \mathcal{A}.$$

Definition. Denote by $\mathcal{X}$ the associative algebra generated by all operators of left and right multiplications by the generators x_i. This algebra is called the algebra of *local operators*.

Definition. A non-abelian Poisson bracket is a bracket of the form

$$\{f,\, g\} = \langle \mathrm{grad}_{\mathbf{x}}\, f,\; \Theta(\mathrm{grad}_{\mathbf{x}}\, g)\rangle, \qquad f, g \in \mathcal{T}, \tag{12}$$

where $\Theta \in \mathcal{X} \otimes \mathfrak{gl}_N$, that satisfies the conditions

$$\{f, g\} + \{g, f\} \sim 0, \tag{13}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \sim 0. \tag{14}$$

Here ~ 0 means equality to zero in $\mathcal{T}$.

Let $H(\mathbf{x}) \in \mathcal{A}$, where $\mathbf{x} = (x_1, \ldots, x_N)$. Then $\mathrm{grad}_{\mathbf{x}}(H) \in \mathcal{A}^N$ is a vector

$$\mathrm{grad}_{\mathbf{x}}\,(H) = \Big(\mathrm{grad}_{x_1}(H), \ldots, \mathrm{grad}_{x_N}(H)\Big),$$

the components of which are uniquely defined by the formula

$$\frac{d}{d\epsilon} H(x_1, \ldots,\, x_k + \epsilon\, \delta_k,\, \ldots, x_N)|_{\epsilon=0} \sim\; \delta_k\, \mathrm{grad}_{x_k}\Big(\, H(\mathbf{x})\Big)\,,$$

where δ_i are additional non-abelian variables.

Example. Let $\mathcal{A} = \acute{[}u, v]$. Let us find the gradient of the polynomial $H = u^2v^2 - uvuu$. We have

$$\frac{d}{d\epsilon} H(u + \epsilon\,\delta_1,\ v)|_{\epsilon=0} =$$

$$\delta_1 uv^2 + u\delta_1 v^2 - \delta_1 vuv - uv\delta_1 v \sim \delta_1\,(uv^2 + v^2u - 2vuv).$$

Therefore, $\mathrm{grad}_u(H) = uv^2 + v^2u - 2vuv = [v,\ [v, u]]$. Similarly, $\mathrm{grad}_v(H) = vu^2 + u^2v - 2uvu = [u,\ [u, v]]$ and

$$\mathrm{grad}_{\mathbf{x}}\,(H) = \Big([v,\ [v, u]],\quad [u,\ [u, v]]\Big),$$

where $\mathbf{x} = (u,\ v)$.

Lemma. If $f \in [\mathcal{A},\ \mathcal{A}]$, then $\mathrm{grad}_{\mathbf{x}}(f) = 0$.

From this lemma it follows that the mapping $\mathrm{grad}_{\mathbf{x}} : \mathcal{T} \to \mathcal{A}^N$ is well defined and the formula (12) defines a bracket on the vector space $\mathcal{T}$.

For any element $f \in \mathcal{A}$, or its equivalence class in $\mathcal{T}$, we denote by f_i the components of its gradient. Sometimes f_i is denoted by $\frac{\partial f}{\partial x_i}$, and $\frac{\partial}{\partial x_i}$ is called "non-abelian partial derivative".

Proposition. For any element $f \in \mathcal{A}$ the identity

$$\sum_{i=1}^{N} [f_i,\, x_i] = 0 \tag{15}$$

is fulfilled.

Any non-abelian Hamiltonian derivation (or vector field) on $\mathcal{A}$ has the form

$$\frac{d\,\mathbf{x}}{d\,t} = \Theta\Big(\mathrm{grad}_{\mathbf{x}} H\Big), \tag{16}$$

where $H(\mathbf{x}) \in \mathcal{A}/[\mathcal{A},\, \mathcal{A}]$ is the Hamiltonian, and Θ is the "Hamiltonian operator" or the "Poisson tensor". According to the definition, Θ is an $N \times N$ matrix with entries being local operators.

Question. Suppose we have two Hamiltonian derivations D_t and D_τ defined by

$$\frac{d\,\mathbf{x}}{d\,t} = \Theta\Big(\mathrm{grad}_{\mathbf{x}} H_1\Big), \tag{17}$$

and

$$\frac{d\,\mathbf{x}}{d\,\tau} = \Theta\Big(\mathrm{grad}_{\mathbf{x}} H_2\Big) \tag{18}$$

Is it true that the commutator $[D_t, D_\tau]$ of these derivations is the Hamiltonian derivation corresponding to $\{H_1,\, H_2\}$?

Linear non-abelian Poisson brackets are given by Hamiltonian operators with the entries of the form

$$\Theta_{ij} = b^k_{ij}\, R_{x_k} + \bar{b}^k_{ij}\, L_{x_k}, \tag{19}$$

where

$$\bar{b}^k_{ij} = -b^k_{ji}. \tag{20}$$

For quadratic non-abelian Poisson brackets, we have

$$\Theta_{i,j} = a^{pq}_{ij} L_{x_p} L_{x_q} + \bar{a}^{pq}_{ij} R_{x_p} R_{x_q} + r^{pq}_{ij} L_{x_p} R_{x_q}, \tag{21}$$

where

$$\bar{a}^{pq}_{ij} = -a^{qp}_{ji}, \qquad r^{pq}_{ij} = -r^{qp}_{ji}. \tag{22}$$

If these constants satisfy conditions (5)–(10) then the corresponding non-abelian brackets satisfy (13), (14).

Any non-abelian Poisson brackets can be extended to the entries of the matrices $x_1, \cdots, x_N$ as follows. We have

$$x^j_{i,\alpha} = \operatorname{tr}(e^i_j x_\alpha), \qquad x^{j'}_{i',\beta} = \operatorname{tr}(e^{i'}_{j'} x_\beta),$$

where e^i_j denotes the matrix units. Set

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = \operatorname{tr}(e^i_j\, \Theta_{\alpha,\beta}(e^{i'}_{j'})). \tag{23}$$

In the linear case (19), (20) we find that

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = b^\gamma_{\alpha,\beta} x^{j'}_{i,\gamma} \delta^j_{i'} - b^\gamma_{\beta,\alpha} x^j_{i',\gamma} \delta^{j'}_i,$$

that is, the extended Poisson bracket coincides with (3).

In the same way, any non-abelian Poisson brackets can be extended to matrix entries using the formula (23). However, the corresponding trace brackets *not always satisfy the Jacobi identity* !

Non-abelian Poisson brackets on the projective space

An usual (scalar) Poisson structure on an affine space $\mathbb{A}^N$ over $\mathbb{C}$ has the form

$$\{f, g\} = \sum_{1 \le i,j \le N} P_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{24}$$

where $x_1, ..., x_N$ are coordinates on $\mathbb{A}^n$ and $P_{i,j} \in \mathbb{C}[x_1, ..., x_N]$ are fixed polynomials. The formula (24) should define a Lie algebra structure on the space of polynomials in $x_1, ..., x_N$.

Which of these Poisson structures can be descended to $\mathbb{C}P^{N-1}$? In fact, $\mathbb{C}P^{N-1} = \mathbb{A}^N/\mathbb{C}^*$, where the group $\mathbb{C}^*$ acts on $\mathbb{A}^n$ by dilatations $x_i \mapsto a x_i$. The bracket (24) should be invariant with respect to this action which means that $P_{i,j}$ have to be homogeneous quadratic polynomials and the formula (24) takes the form

$$\{f, g\} = \sum_{1 \le i,j,a,b \le N} r^{a,b}_{i,j} x_a x_b \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}. \tag{25}$$

To descend this Poisson structure to $\mathbb{C}P^{N-1}$ we introduce affine coordinates $u_i = \frac{x_i}{x_N},\ i = 1...N-1$. If f, g are functions in $u_1, ..., u_{N-1}$, then, after the change of variables, the formula (25) can be rewritten as

$$\{f, g\} = \sum_{\substack{1\le i,j\le N-1,\\ 1\le a,b\le N}} (r^{a,b}_{i,j} u_a u_b + r^{a,b}_{j,N} u_a u_b u_i + r^{a,b}_{i,N} u_a u_b u_j) \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j},$$

where $u_N = 1$.

Example. The simplest example of elliptic homogeneous Poisson brackets is given by

$$\{x_1, x_2\} = t\, x_1 x_2 + x_3^2, \qquad \{x_2, x_3\} = t\, x_2 x_3 + x_1^2,$$

$$\{x_3, x_1\} = t\, x_1 x_3 + x_2^2,$$

where $t \in \mathbb{C}$ is a parameter. In the affine coordinates $u_1 = \frac{x_1}{x_3},\ u_2 = \frac{x_2}{x_3}$ this Poisson structure has the form

$$\{u_1,\, u_2\} = u_1^3 + u_2^3 + 3t\, u_1 u_2 + 1.$$

The first example of an elliptic Poisson bracket with 4 generators was constructed by E. Sklyanin.

A. Odesskii and B. Feygin constructed a wide class of elliptic brackets of the form (25) named $q_{n,k}(\tau)$. Here $n, k \in \mathbb{Z}$, $1 \leq k < n$ and n, k are coprime.

The bracket $q_{n,k}(\tau)$ admits a discrete group of automorphisms acting on generators $x_1, \ldots, x_n$ by $x_i \mapsto \varepsilon^i x_i$ and $x_i \mapsto x_{i+1}$, where ε is a primitive n-th root of unity.

The Poisson brackets $q_{n,n-1}(\tau)$ are trivial.

An explicit formula for the coefficients of $q_{n,k}(\tau)$ can be written in terms of theta constants.

Explicitly, the Poisson brackets in $q_{n,k}(\tau)$ are the following:

$$\{x_i, x_j\} = \left(\frac{\theta'_{j-i}(0)}{\theta_{j-i}(0)} + \frac{\theta'_{k(j-i)}(0)}{\theta_{k(j-i)}(0)} \right) x_i x_j + \sum_{r \neq 0, j-i} \frac{\theta'_0(0)\theta_{j-i+r(k-1)}(0)}{\theta_{kr}(0)\theta_{j-i-r}(0)} x_{j-r} \quad (26)$$

Notice that the algebra $Q_{n,k}(\eta, \tau)$ and the corresponding Poisson algebra $q_{n,k}(\tau)$ both admit the same discrete group of automorphisms. Namely, a central extension of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, the so-called discrete Heisenberg group, acts on generators by $x_i \mapsto \varepsilon^i x_i$ and $x_i \mapsto x_{i+1}$, where ε is a primitive n-th root of unity.

If we want to construct a Poisson bracket on $\mathbb{C}P^{N-1}$ starting from (25), then Jacobi identity for $\{f, g\}$ is sufficient but *not necessary condition.* !

Indeed, we need Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

for homogeneous f, g, h only. But any homogeneous function satisfies the Euler identity

$$x_1 \frac{\partial f}{\partial x_1} + ... + x_N \frac{\partial f}{\partial x_N} = 0. \qquad (27)$$

Therefore, (25) descends to a Poisson structure on $\mathbb{C}P^{N-1}$ if Jacobi identity $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ is satisfied modulo (27) and similar identities for g, h.

This observation turns to be crucial for generalization of elliptic Poisson brackets to the non-abelian case.

On non-Abelian brackets

Following M. Kontsevich, we consider a free associative algebra

$$A = \mathbb{C}[x_1, ..., x_N]$$

as a “noncommutative affine space". The commutant

$$\mathcal{T} = A/[A, A]$$

is “the space of functions on the noncommutative affine space".

Brackets should be defined on $\mathcal{T}$ and satisfy the identites

$$\{f, g\} = -\{g, f\}, \qquad \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

where $f, g, h \in \mathcal{T} = A/[A, A]$.

Remark. There is no any Leibniz rule, since $\mathcal{T}$ is a vector space, not an algebra.

By definition, a non-abelian polynomial Poisson structure on an affine space has the form

$$\{f, g\} = tr\Big(\sum_{\substack{1\le i,j\le N,\\ 1\le s\le K}} P_{i,j,s} \frac{\partial f}{\partial x_i} Q_{i,j,s} \frac{\partial g}{\partial x_j} \Big) \tag{28}$$

for some K. Here $P_{i,j,s}, Q_{i,j,s}$ are fixed elements of the free algebra A; $f, g \in \mathcal{T} = A/[A, A]$; $tr: A \to \mathcal{T}$ is a natural map and $\frac{\partial}{\partial x_i}: \mathcal{T} \to A$ are the non-abelian partial derivatives (they are not vector fields!).

The formula (28) should define a Lie algebra structure on $\mathcal{T}$.

Before we wrote non-abelian Poisson brackets in the form (12):

$$\{f,\, g\} = tr\Big(\sum_{1 \le i,j \le n} \frac{\partial f}{\partial x_i} \Theta_{i,j}\Big(\frac{\partial g}{\partial x_j}\Big)\Big), \tag{29}$$

where

$$\Theta_{i,j} = \sum_{1 \le s \le N} P_{i,j,s} \otimes Q_{i,j,s} \in \mathcal{A} \otimes \mathcal{A}^{op}.$$

We assume that $\mathcal{A} \otimes \mathcal{A}^{op}$ acts on $\mathcal{A}$ in the standard way: $a \otimes b(c) = acb$.

If (29) is non-abelian Poisson structure, then $\Theta_{i,j}$ is an analog of the Poisson tensor for its abelianization.

We assume that projective objects should be invariant with respect to the change of variables

$$x_i \mapsto a x_i, \qquad i = 1, ..., N, \tag{30}$$

where a is an auxiliary *noncommutative* variable. We consider the following non-abelian generalization of the brackets (25):

$$\{f, g\} = tr\Big(\sum_{1 \le i,j,a,b \le N} r^{a,b}_{i,j} x_a \frac{\partial f}{\partial x_i} x_b \frac{\partial g}{\partial x_j} \Big). \tag{31}$$

It turns out that the bracket (31) is invariant with respect to (30).

To descend the non-abelian Poisson structure (31) to $\mathbb{C}P^{N-1}$, we introduce affine coordinates

$$u_i = x_N^{-1} x_i, \qquad i = 1, ..., N-1.$$

It is clear that $u_1, ..., u_{N-1}$ are invariant with respect to transformations (30).

If f, g are noncommutative polynomials in $u_1, ..., u_{N-1}$, then, after the change of variables, the formula (31) can be rewritten as

$$\{f,g\} = \sum_{\substack{1\le i,j\le N-1,\\ 1\le a,b\le N}} tr\Big(r^{a,b}_{i,j} u_a \frac{\partial f}{\partial u_i} u_b \frac{\partial g}{\partial u_j} - r^{a,b}_{N,j} u_a u_i \frac{\partial f}{\partial u_i} u_b \frac{\partial g}{\partial u_j} - r^{a,b}_{i,N} u_a \frac{\partial f}{\partial u_i} u_b u_j \frac{\partial g}{\partial u_j} + r^{a,b}_{N,N} u_a u_i \frac{\partial f}{\partial u_i} u_b u_j \frac{\partial g}{\partial u_j} \Big), \tag{32}$$

where we assume that $u_N = 1$.

It turns out (contrary to the commutative case) that not all non-abelian Poisson structures on $\mathbb{C}P^{N-1}$ can be obtained in this way from non-abelian Poisson structures on $\mathbb{A}^N$.

Example. The non-abelian analog of $q_{3,1}(\tau)$ has the form

$$\{f,g\} = \sum_{i\in\mathbb{Z}/3\mathbb{Z}} tr\,\Big(\frac{1}{2}t\frac{\partial f}{\partial x_{i+1}}x_{i+1}\frac{\partial g}{\partial x_{i+2}}x_{i+2}+ \frac{1}{2}t\frac{\partial f}{\partial x_{i+1}}x_{i+2}\frac{\partial g}{\partial x_{i+2}}x_{i+1} + \frac{\partial f}{\partial x_{i+1}}x_i\frac{\partial g}{\partial x_{i+2}}x_i- \frac{1}{2}t\frac{\partial f}{\partial x_{i+2}}x_{i+1}\frac{\partial g}{\partial x_{i+1}}x_{i+2} - \frac{1}{2}t\frac{\partial f}{\partial x_{i+2}}x_{i+2}\frac{\partial g}{\partial x_{i+1}}x_{i+1}- \frac{\partial f}{\partial x_{i+2}}x_i\frac{\partial g}{\partial x_{i+1}}x_i\Big).$$

This bracket does not satisfy the Jacobi identity.

The corresponding Poisson structure on $\mathbb{C}P^2$ is given by

$$\{f, g\} = tr\left(\sum_{1 \leq i,j \leq 2} \frac{\partial f}{\partial u_i} \Theta_{i,j}\left(\frac{\partial g}{\partial u_j}\right)\right),$$

where

$$\Theta_{1,1} = -u_1 u_2 \otimes u_2 + u_2 \otimes u_1 u_2, \qquad \Theta_{2,2} = u_2 u_1 \otimes u_1 - u_1 \otimes u_2 u_1.$$

$$\Theta_{1,2} = u_1^2 \otimes u_1 + u_2 \otimes u_2^2 + \frac{t}{4} u_2 \otimes u_1 + \frac{t}{4} u_1 \otimes u_2 + 1,$$

$$\Theta_{2,1} = -u_1 \otimes u_1^2 - u_2^2 \otimes u_2 - \frac{t}{4} u_2 \otimes u_1 - \frac{t}{4} u_1 \otimes u_2 - 1.$$

This bracket satisfies the Jacobi identity.

Non-abelian Poisson structures on $\mathbb{C}P^{N-1}$

Let us generalize the usual definition to the noncommutative case. We embed our free associative algebra $A=\mathbb{C}[x_1,...,x_N]$ into the algebra of non-abelian Laurent polynomials $\hat{A}=\mathbb{C}[x_1,...,x_N,x_1^{-1},...,x_N^{-1},a,a^{-1}]$, where a is an additional auxiliary generator.

Let $\hat{\mathcal{T}}=\hat{A}/[\hat{A},\hat{A}]$. We define a homomorphism $f\mapsto f^a,\ f\in\hat{A}$ of the algebra $\hat{A}$ to itself by $x_i\mapsto ax_i,\ i=1...,N$ and $a\mapsto a$.

Definition. An element $f\in\hat{A}$ is called homogeneous if $f^a=f$. In this case an element $tr(f)\in\hat{\mathcal{T}}$ is also called homogeneous.

We consider $x_1, ..., x_N \in A$ as homogeneous coordinates on a noncommutative projective space $\mathbb{C}P^{N-1}$. Homogeneous elements in $\hat{F}$ are considered as functions on $\mathbb{C}P^{N-1}$.

Proposition. Let $f \in \hat{\mathcal{T}}$ be a homogeneous element. Then the following identities hold:

$$x_1 \frac{\partial f}{\partial x_1} + ... + x_N \frac{\partial f}{\partial x_N} = 0, \quad \frac{\partial f}{\partial x_1} x_1 + ... + \frac{\partial f}{\partial x_N} x_N = 0. \tag{33}$$

Define a homogeneous non-abelian bivector field

$$\nu(f, g) = tr\Big(\sum_{i,j,r \in \mathbb{Z}/N\mathbb{Z}} c_{i-j,r} \frac{\partial f}{\partial x_i} x_{i-r} \frac{\partial g}{\partial x_j} x_{j+r} \Big), \tag{34}$$

where

$$c_{i,r} = \frac{\theta_0'(0)\theta_{i+r(k-1)}(0)}{\theta_{kr}(0)\theta_{i-r}(0)}, \qquad r \neq 0,\ i, \tag{35}$$

$$c_{0,0} = 0, \qquad c_{i,0} = \frac{\theta_i'(0)}{\theta_i(0)}, \qquad c_{i,i} = \frac{\theta_{ki}'(0)}{\theta_{ki}(0)}.$$

Remark. The non-abelian bivector field defined by (34) does not give an affine non-abelian Poisson structure. However, its abelianization satisfies the Jacobi identity and coincides with the Poisson algebra $q_{N,k}(\tau)$.

Remark. In the case $k = N - 1$ the non-abelian bivector field (34) is nonzero.

The following statement is the main result:

Theorem. For any coprime N and k the formula (34) defines a non-abelian Poisson structure of the form (32) on $\mathbb{C}P^{N-1}$.

Theta functions of one variable

Define a holomorphic function $\theta(z)$ by

$$\theta(z) = \sum_{\alpha\in\mathbb{Z}} (-1)^{\alpha} e^{2\pi i\left(\alpha z+\frac{\alpha(\alpha-1)}{2}\tau\right)}$$

It is clear that

$$\theta(z+1) = \theta(z), \quad \theta(z+\tau) = -e^{-2\pi i z}\theta(z), \quad \theta(-z) = -e^{-2\pi i z}\theta(z).$$

Define the so called theta functions with characteristics by

$$\theta_{\alpha}(z) = \theta\left(z+\frac{\alpha}{N}\tau\right)\theta\left(z+\frac{1}{N}+\frac{\alpha}{N}\tau\right)\ldots\theta\left(z+\frac{N-1}{N}+\frac{\alpha}{N}\tau\right)\times e^{\pi i((2\alpha-N)z-\frac{\alpha}{N}+\frac{\alpha(\alpha-N)}{N}\tau)}$$

One can check that $\theta_{\alpha+N}(z) = \theta_{\alpha}(z)$ so we can consider α as an element in $\mathbb{Z}/N\mathbb{Z}$.

One can also check that

$$\theta_{\alpha}(z+1) = (-1)^{N}\theta_{\alpha}(z), \quad \theta_{\alpha}(z+\tau) = -e^{-2\pi i N(z+\frac{1}{2}\tau)}\theta_{\alpha}(z)$$

and

$$\theta_{\alpha}(-z) = -e^{-\frac{2\pi i\alpha}{N}}\theta_{-\alpha}(z). \tag{36}$$

The following identities can be proved in a standard way.

Lemma. Let $N = 3$. Then

$$\theta_0(z)^3 + \theta_1(z)^3 + \theta_2(z)^3 + 3t\,\theta_0(z)\theta_1(z)\theta_2(z) = 0,$$

where t is a certain function in τ.

Lemma. Let $N > 3$. Then

$$\theta_{j+k}(0)\theta_{j-k}(0)\theta_{l+i}(z)\theta_{l-i}(z) + \theta_{k+i}(0)\theta_{k-i}(0)\theta_{l+j}(z)\theta_{l-j}(z) +$$

$$\theta_{i+j}(0)\theta_{i-j}(0)\theta_{l+k}(z)\theta_{l-k}(z) = 0$$

for arbitrary $i, j, k, l \in \mathbb{Z}/N\mathbb{Z}$. In particular, for $z = 0$ we have

$$\theta_{j+k}(0)\theta_{j-k}(0)\theta_{l+i}(0)\theta_{l-i}(0) + \theta_{k+i}(0)\theta_{k-i}(0)\theta_{l+j}(0)\theta_{l-j}(0) +$$

$$\theta_{i+j}(0)\theta_{i-j}(0)\theta_{l+k}(0)\theta_{l-k}(0) = 0.$$