Hierarchies of compatible maps and integrable difference systems[1]

Pavlos Kassotakis

Integrable systems and related topics: Yaroslavl online seminar,
27 April 2022

[1]Based on: P.K. and T. Kouloukas, J.Phys.A. 55 (17), 175203, 2022
and P.K. arXiv:2202.03412, 2022

Outline

- Introduction
  - Compatible maps/Yang-Baxter maps
- An important question. Non-Abelian compatible maps/Yang-Baxter maps
- Hierarchies of hierarchies of non-Abelian compatible maps/Yang-Baxter maps
- Non-Abelian integrable difference systems

INTRODUCTION

Integrability manifested by compatibility of maps

- Let $\mathbb{X}$ be any set

Let $F : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{X} \times \mathbb{X}$, be a map and F_{ij} $i < j \in \{1, 2, 3\}$, be the maps that act as F on the $i-$th and $j-$th factor of $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$. We denote these maps in components as follows

$$F_{ij} : (u^i, u^j, u^k) \mapsto (u^i_j, u^j_i, u^k) = (u^i_j(u^i, u^j), u^j_i(u^i, u^j), u^k), \quad i \neq j \neq k \neq i \in \{1, 2, 3\}.$$

3D-compatible map (Adler,Bobenko,Suris 2004)

A map $F : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ will be called *3D-compatible map* if it holds $u^i_{jk} = u^i_{kj}$ i.e.

$$u^i_j \left(u^i_k(u^i, u^k), u^j_k(u^j, u^k) \right) = u^i_k \left(u^i_j(u^i, u^j), u^k_j(u^k, u^j) \right), \quad i \neq j \neq k \neq i \in \{1, 2, 3\}.$$

3D compatibility

Consider the maps

$$F_{12} : (u^1, u^2, u^3) \mapsto (u^1_2, u^2_1, u^3),$$

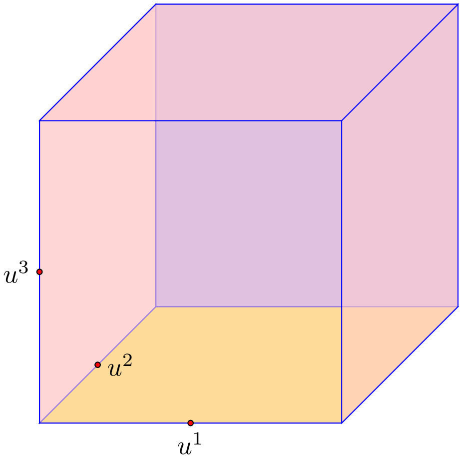
$$F_{23} : (u^1, u^2, u^3) \mapsto (u^1, u^2_3, u^3_2),$$

$$F_{13} : (u^1, u^2, u^3) \mapsto (u^1_3, u^2, u^3_1),$$

This system is called 3D-compatible, if it holds

$$u^1_{23} = u^1_{32},\ u^2_{13} = u^2_{31},\ u^3_{12} = u^3_{21}$$

for any choice of initial data u^1, u^2, u^3.



- **Compatibility**

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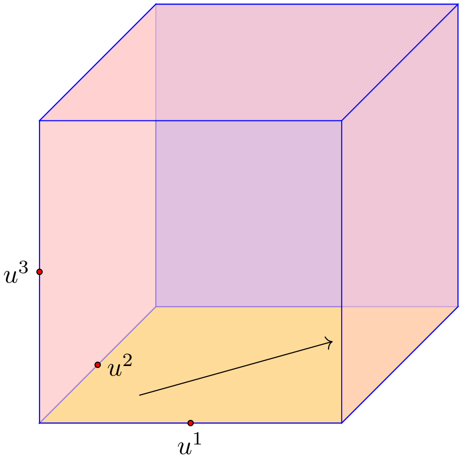

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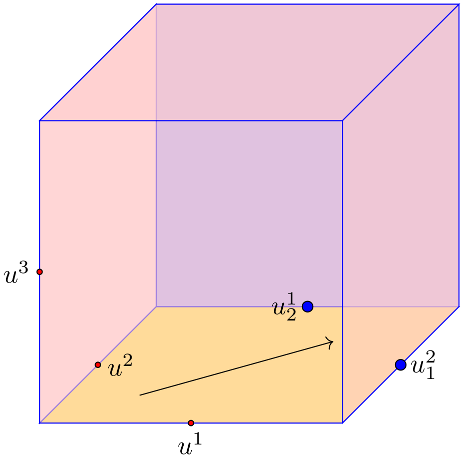

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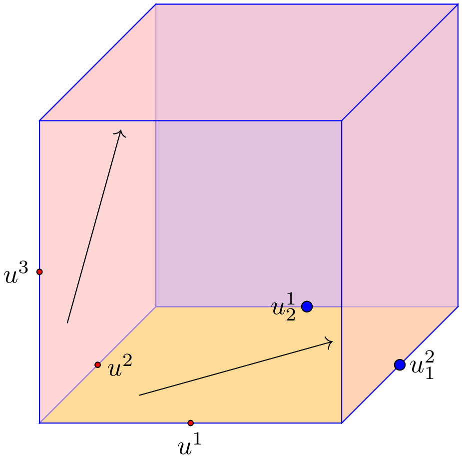

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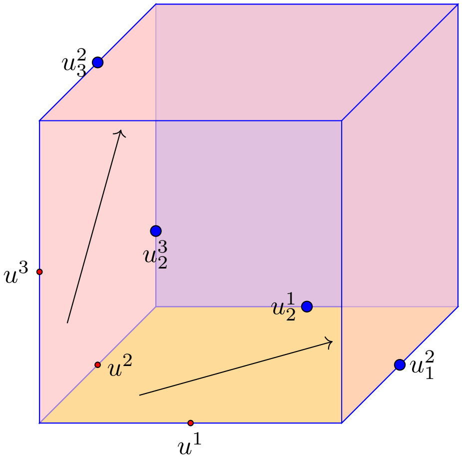

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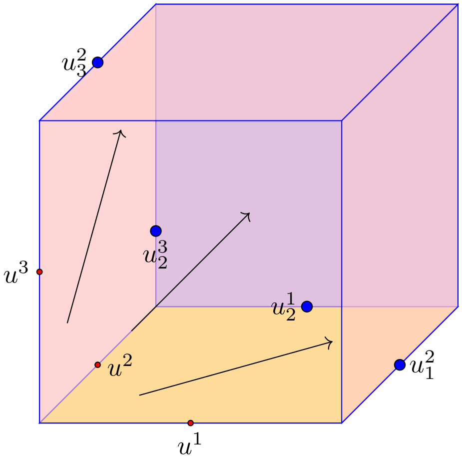

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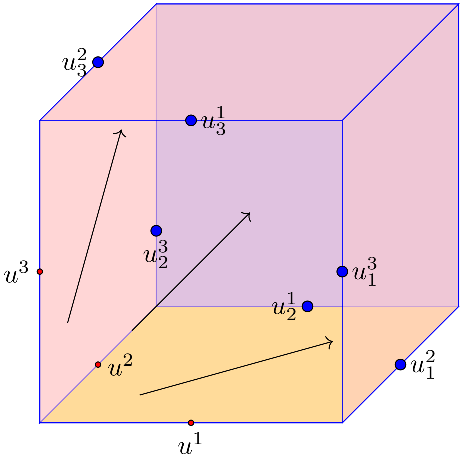

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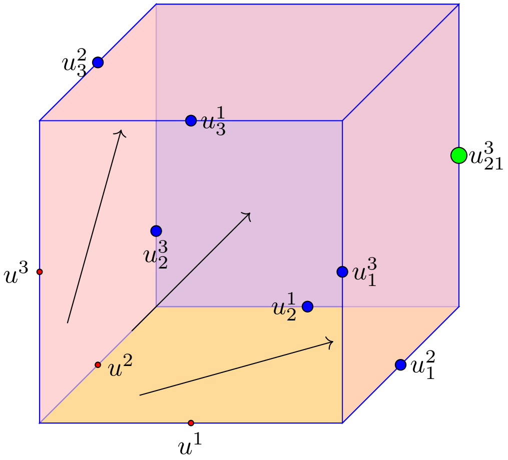

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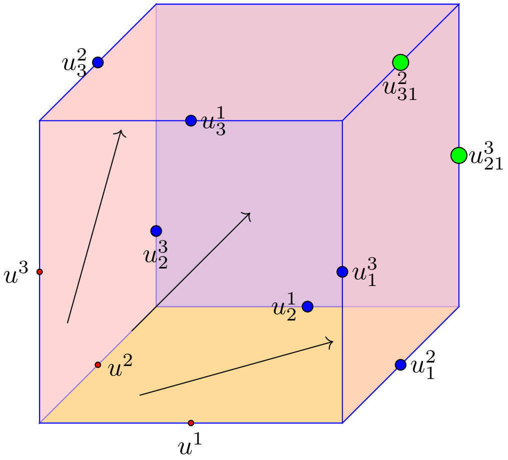

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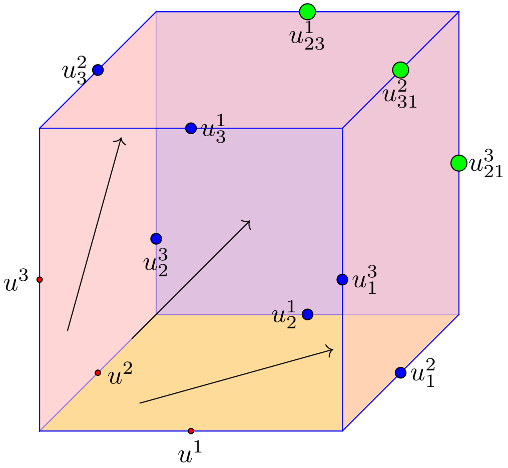


- **Compatibility**

$$u^1_{23} = u^1_{32}, \quad u^2_{13} = u^2_{31}, \quad u^3_{12} = u^1_{21}$$

Alternatively we can use the notion of *Yang-Baxter maps*

Yang-Baxter map (Sklyanin (1988), Drinfeld (1990))

A map $R: \mathbb{X} \times \mathbb{X} \ni (u, v) \mapsto (x, y) = (x(u, v), y(u, v)) \in \mathbb{X} \times \mathbb{X}$, will be called *Yang-Baxter map* if it satisfies the *Yang-Baxter relation*

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},$$

where $R_{i,j}$ $i, j \in \{1, 2, 3\}$, denotes the action of the map R on the $i-$th and the $j-$th factor of $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$, i.e.

$$R_{12} : (u, v, w) \mapsto (x(u, v), y(u, v), w),$$

$$R_{13} : (u, v, w) \mapsto (x(u, w), v, z(u, w)),$$

and

$$R_{23} : (u, v, w) \mapsto (u, y(v, w), z(v, w)).$$

- **Drinfeld (1990)** Proposed the term *Set theoretical solutions of the quantum Yang-Baxter equation*. **Buchstaber (1998)** Proposed the term *Yang-Baxter transformations*. **Veselov (2003)** *Yang-Baxter maps*.

Or we can use the notion of the *Braid maps*

Braid/Trefoil maps

A map $S : \mathbb{X} \times \mathbb{X} \ni (u, v) \mapsto (x, y) = (x(u, v), y(u, v)) \in \mathbb{X} \times \mathbb{X}$, will be called *Braid map* if it satisfies the *Braid relation*

$$S_{12} \circ S_{23} \circ S_{12} = S_{23} \circ S_{12} \circ S_{23},$$

where $S_{i,j}$ $i, j \in \{1, 2, 3\}$, denotes the action of the map S on the $i-$th and the $j-$th factor of $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$.

- The following are equivalent:
  (i) R is a YB map $\quad$ (ii) $\tau\, R$ is a braid map
  (iii) $\tau\, R\, \tau$ is a YB map $\quad$ (iv) $R\, \tau$ is a braid map

- The Artin's Braid group $\mathbb{B}_n$:

$$\mathbb{B}_n =< b_1, \ldots, b_{n-1} | b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, b_i b_j = b_j b_i >$$

$$1 \leq i \leq n-2, \quad |i-j| \geq 2.$$

Definitions Summary

Equivalency of maps

The maps $R : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ and $\widehat{R} : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ will be called YB equivalent if it exists a bijection $\kappa : \mathbb{X} \to \mathbb{X}$ such that

$$(\kappa \times \kappa)\, R = \widehat{R}\, (\kappa \times \kappa).$$

Symmetry

A bijection $\phi : \mathbb{X} \to \mathbb{X}$ will be called *symmetry* of the map $R : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$, if $(\phi \times \phi)\, R = R\, (\phi \times \phi)$.

- Yang-Baxter map

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$$

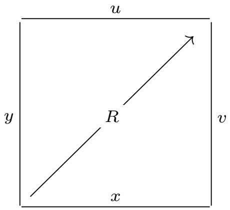
- 3D-compatible maps

$$u^i_j \left(u^i_k(u^i, u^k), u^j_k(u^j, u^k) \right) = u^i_k \left(u^i_j(u^i, u^j), u^k_j(u^k, u^j) \right), \quad i \neq j \neq k \neq i \in \{1, 2, 3\}.$$

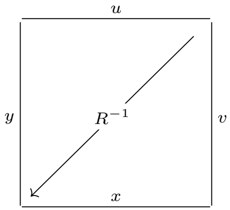
- YB equivalency respects the Yang-Baxter property as well as 3D-compatibility.

Quadrirational maps (Etingof 2003)

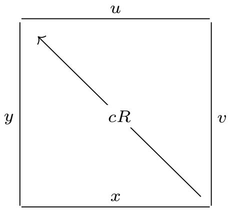
A map $R : \mathbb{X} \times \mathbb{X} \ni (u, v) \mapsto (x, y) \in \mathbb{X} \times \mathbb{X}$ will be called quadrirational, if both the map R and the so called *companion map* $cR : \mathbb{X} \times \mathbb{X} \ni (x, v) \mapsto (u, y) \in \mathbb{X} \times \mathbb{X}$, are birational maps.



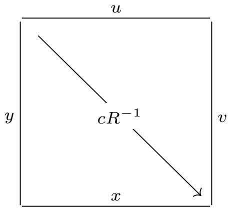
The map R



The inverse map R^{-1}



The companion map cR



The inverse of the companion map cR^{-1}

• The companion map of a quadrirational Yang-Baxter map is a 3D-compatible map. The converse also holds i.e. the companion map of a quadrirational 3D-compatible map is a Yang-Baxter map.

Lax matrices (Suris and Veselov 2003, Nijhoff 2002)

The matrix $L(x;\lambda)$

1. is called a Lax matrix of the Yang-Baxter map $R:(u,v)\mapsto(x,y)$, if the relation $R(u,v)=(x,y)$ implies that

$$L(u;\lambda)L(v;\lambda)=L(y;\lambda)L(x;\lambda)$$

for all λ;

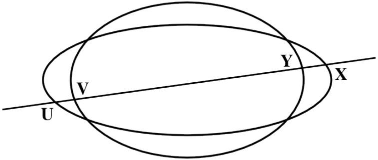
2. is called a Lax matrix of the companion map $cR:(x,v)\mapsto(u,y)$, if the relation $cR(x,v)=(u,y)$ implies that

$$L(u;\lambda)L(v;\lambda)=L(y;\lambda)L(x;\lambda)$$

for all λ. $L(x;\lambda)$ is called a strong Lax matrix of cR if the converse also holds.

The $F-$list of quadrirational Yang-Baxter maps[3]

The five maps F_I–F_V correspond to the five possible types of intersection of two conics



A quadrirational map on a pair of conics

- four simple intersection points
- two simple intersection points and one point of tangency
- two points of tangency
- one simple intersection point and one point of the second order tangency;
- one point of the third order tangency.

[3] **Adler, Bobenko, Suris (2004)**

The $F-$list of quadrirational Yang-Baxter maps

The Yang-Baxter maps of the $F-$list, explicitly reads:
$\mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u,v) \mapsto (x,y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ where:

$$x = \alpha v P, \quad y = \beta u P, \qquad P = \frac{(1-q)u + q - p + (p-1)v}{q(1-p)u + (p-q)uv + p(q-1)v}, \qquad (F_I),$$

$$x = \alpha v P, \quad y = \beta u P, \qquad P = \frac{u - v + q - p}{qu - pv}, \qquad (F_{II}),$$

$$x = \frac{v}{\alpha} P, \quad y = \frac{u}{\beta} P, \qquad P = \frac{pu - qv}{u - v}, \qquad (F_{III}),$$

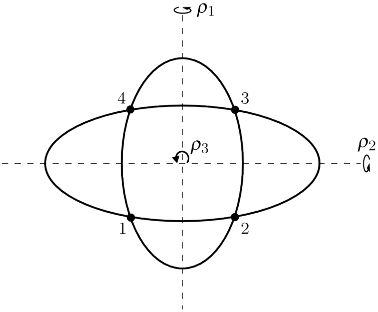
$$x = vP, \quad y = uP, \qquad P = 1 + \frac{q-p}{u-v}, \qquad (F_{IV}),$$

$$x = v + P, \quad y = u + P, \qquad P = \frac{p-q}{u-v}, \qquad (F_V),$$

The maps above are depending on 2 complex parameters p, q. The parameter p is associated with the first factor of the cartesian product $\mathbb{CP}^1 \times \mathbb{CP}^1$, whereas the parameter q with the second factor.

Geometric symmetries and Yang-Baxter maps

The maps F_I–$F_V : (u, v) \mapsto (x, y)$ are just the coordinate representations of the map $\mathcal{R} : (\mathbf{X}, \mathbf{Y}) \mapsto (\mathbf{U}, \mathbf{V})$ in some special rational parametrization of the conics, α and β being the cross-ratios of four points.



$$Q1 : w_2(w_2 - 1) = \alpha w_1(w_1 - 1) \quad Q_2 : w_2(w_2 - 1) = \beta w_1(w_1 - 1),$$

- It has the symmetry group $K=\{id,\rho_1,\rho_2,\rho_3\}$, where

$$\rho_1(w_1,w_2)=(1-w_1,w_2),\quad \rho_2(w_1,w_2)=(w_1,1-w_2),\quad \rho_3(w_1,w_2)=(1-w_1,1-w_2).$$

Proposition (Papageorgiou et.al. 2010)

Let $\phi:\mathbb{X}\to\mathbb{X}$ a symmetry of the Yang-Baxter map $R:\mathbb{X}\times\mathbb{X}\to\mathbb{X}\times\mathbb{X}$. Then the map

$$\widehat{R}=(\phi^{-1}\times id)\,R\,(id\times\phi),$$

is also a Yang-Baxter map.

Clearly the Yang-Baxter maps R and $\widehat{R}$ are not YB equivalent.

- From geometric symmetries of the $F-$list the $H-$list[4] is obtained

[4] **Papageorgiou, Suris, Tongas, Veselov (2010)**

The $H-$list of quadrirational Yang-Baxter maps

The Yang-Baxter maps of the $H-$list, explicitly reads:
$\mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u,v) \mapsto (x,y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ where:

$$x = vQ, \quad y = uQ^{-1}, \qquad Q = \frac{(p-1)uv + (q-p)u + p(1-q)}{(q-1)uv + (p-q)v + q(1-p)}, \qquad (H_I)$$

$$x = v + Q, \quad y = u - Q, \qquad Q = \frac{(p-q)uv}{qu + pv - pq}, \qquad (H_{II})$$

$$x = \frac{v}{\alpha}Q, \quad y = \frac{u}{\beta}Q, \qquad Q = \frac{pu + qv}{u + v}, \qquad (H_{III}^A)$$

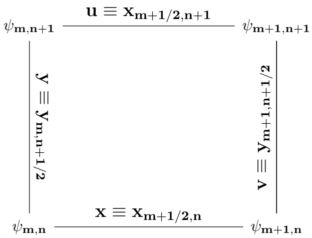
$$x = vQ, \quad y = uQ^{-1}, \qquad Q = \frac{1 + quv}{1 + puv}, \qquad (H_{III}^B)$$

$$x = v - P, \quad y = u + P, \qquad Q = \frac{p-q}{u+v}, \qquad (H_V)$$

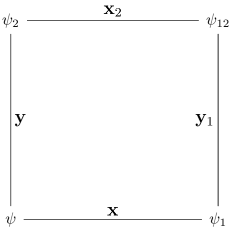
The maps above are depending on 2 complex parameters p, q. p is associated with the first factor of the cartesian product whereas q with the second factor.

Maps as difference systems

$$R : (x, y) \mapsto (u, v)$$



(a) Descriptive notation



(b) Compendious notation

Variables assigned on vertices and edges of an elementary cell of the $\mathbb{Z}^2$ graph

- **Edge equations**
- **Vertex equations**

Vertex and edge equations are usually related

Classification of integrable vertex-equations. The ABS[7] list

$$p(xx_1+x_2x_{12})-q(xx_2+x_1x_{12})-C\,(xx_{12}+x_1x_2-pq(1+xx_1x_2x_{12}))=0, \qquad (Q4)$$

$$a(xx_1+x_2x_{12})-b(xx_2+x_1x_{12})-c(x_1x_2+xx_{12})-\delta abc=0, \qquad (Q3^\delta)$$

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q)(x+x_1+x_2+x_{12}-D)=0, \qquad (Q2)$$

– –

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta pq(p-q)=0, \qquad (Q1^\delta)$$

$$a(xx_2+x_1x_{12})-b(xx_1+x_2x_{12})-c(1+xx_1x_2x_{12})=0, \qquad (A2)$$

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta pq(p-q)=0, \qquad (A1^\delta)$$

$$p(xx_1+x_2x_{12})-q(xx_2+x_1x_{12})+\delta(p^2-q^2)=0, \qquad (H3^\delta)$$

$$(x-x_{12})(x_1-x_2)-(p-q)(x+x_1+x_2+x_{12}+p+q)=0, \qquad (H2)$$

$$(x-x_{12})(x_1-x_2)+p-q=0. \qquad (H1)$$

- The equations below the red line ARE related to the F or H list
- The equations above the red line ARE NOT related to the F or H list.

$$\delta=0,1,\ a:=\left(p-\frac{1}{p}\right),\ b:=\left(q-\frac{1}{q}\right),\ c:=\left(\frac{p}{q}-\frac{q}{p}\right),\ C:=\frac{pQ-Pq}{1-p^2q^2},\ D:=p^2-pq+q^2,$$

where $P^2=p^4-\gamma p^2+1$, $Q^2=q^4-\gamma q^2+1$, so (p,P) and (q,Q) are points on the elliptic curve $\mathcal{E}=\{(\alpha,A)\in\mathbb{C}^2 : A^2=\alpha^4-\gamma\alpha^2+1\}$, with γ the modulus of $\mathcal{E}$.

[7] Adler, Bobenko, Suris 2003

An important question.

- Can $F-$list and $H-$list of maps be extended in the non-abelian domain?

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- Can $F-$list and $H-$list of maps be extended in the non-abelian domain?
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Motivation

- Hierarchies of scalar maps
- Towards quantization

Non-abelian setting

- Let $\mathbb{A}$ be a free associative algebra over a field $\mathbb{F}$, with multiplicative identity that we denote with 1.

- We consider $\mathrm{X} = \underbrace{\mathbb{A}^{\times} \times \cdots \times \mathbb{A}^{\times}}_{N-\text{times}}$, $N \in \mathbb{N}$, where $\mathbb{A}^{\times}$ denotes the subgroup of elements $w \in \mathbb{A}$ having multiplicative inverse $w^{-1} \in \mathbb{A}$, s.t. $ww^{-1} = w^{-1}w = 1$.

- With $C(\mathbb{D})$ we denote the center of algebra $\mathbb{A}^{\times}$ i.e. a commutative subalgebra of $\mathbb{A}^{\times}$ consisting of invertible elements.

- F.i. $\mathbb{A}^{\times}$ could be a division ring for instance bi-quaternions, or more generally, $\mathbb{A}^{\times}$ could stand for the subgroup of invertible matrices of the algebra $\mathbb{A}$ of $n \times n$ matrices.

$$R : \mathrm{X} \times \mathrm{X} \ni (\mathbf{x}, \mathbf{y}) \to (\mathbf{u}, \mathbf{v}) \in \mathrm{X} \times \mathrm{X}$$

- Note that $\mathbf{x}$ could be a collection of variables $\mathbf{x} = (x^1, \ldots, x^N)$, $N \in \mathbb{N}$.

Non-abelian Yang-Baxter maps

Centrality assumption (Doliwa 2013)

when we refer to the centrality assumption for a map $F:(z^1,z^2,w^1,w^2)\mapsto(\bar z^1,\bar z^2,\bar w^1,\bar w^2)$ we refer to

$$z^2z^1=p\in C(\mathbb{D}),\quad w^2w^1=q\in C(\mathbb{D}),$$

where $C(\mathbb{D})$ the center of the algebra $\mathbb{D}$.

(P.K. and T. Kouloukas 2022)

Provided the centrality assumption, the map

$$c\mathcal{K}_{a,b,c}:(x^1,x^2,v^1,v^2)\mapsto(u^1,u^2,y^1,y^2),$$

where

$$u^1=(b-cv^1)(x^2-v^2)x^1\left(x^1-v^1\right)^{-1}\left(a-cv^2\right)^{-1},$$
$$u^2=(a-cv^2)(x^1-v^1)x^2\left(x^2-v^2\right)^{-1}\left(b-cv^1\right)^{-1},$$
$$y^1=(b-cx^1)(x^2-v^2)v^1\left(x^1-v^1\right)^{-1}\left(a-cx^2\right)^{-1},$$
$$y^2=(a-cx^2)(x^1-v^1)v^2\left(x^2-v^2\right)^{-1}\left(b-cx^1\right)^{-1},$$

with $a,b,c\in C(\mathbb{D})$ and neither a,c nor b,c simultaneously zero,

- has as symmetries the bijections

$$\psi : (z^1, z^2) \mapsto (\frac{b}{a} z^2, \frac{a}{b} z^1), \tag{2}$$

$$\phi : (z^1, z^2) \mapsto \left(\frac{b}{a}(a - cz^2) z^1 (cz^1 - b)^{-1}, \frac{a}{b}(b - cz^1) z^2 (cz^2 - a)^{-1} \right); \tag{3}$$

- has as strong Lax matrix the matrix

$$L(x^1, x^2; \lambda) = \begin{pmatrix} ax^1 - cx^2 x^1 & \lambda(b - cx^1) \\ a - cx^2 & bx^2 - cx^1 x^2 \end{pmatrix}, \tag{4}$$

where the spectral parameter $\lambda \in C(\mathbb{D})$;

- it is a $3D-$compatible quadrirational map and its companion map reads

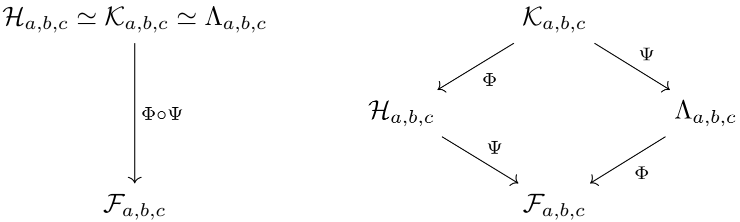
$$\mathcal{K}_{a,b,c} : (u^1, u^2, v^1, v^2) \mapsto (x^1, x^2, y^1, y^2),$$

where

$$x^1 = \left(au^1 + bv^2 - c(v^1 v^2 + u^1 v^2)\right)^{-1} u^1 \left(av^1 + bu^2 - c(v^2 v^1 + u^2 v^1)\right),$$
$$x^2 = \left(bu^2 + av^1 - c(v^2 v^1 + u^2 v^1)\right)^{-1} u^2 \left(bv^2 + au^1 - c(v^1 v^2 + u^1 v^2)\right),$$
$$y^1 = \left(au^1 + bv^2 - c(u^1 u^2 + u^1 v^2)\right) v^1 \left(bu^2 + av^1 - c(u^2 u^1 + u^2 v^1)\right)^{-1},$$
$$y^2 = \left(bu^2 + av^1 - c(u^2 u^1 + u^2 v^1)\right) v^2 \left(au^1 + bv^2 - c(u^1 u^2 + u^1 v^2)\right)^{-1}.$$

- The map $\mathcal{K}_{a,b,c}$ is a Yang-Baxter map.

The non-Abelian $\mathcal{F}$, $\mathcal{H}$, $\mathcal{K}$ and Λ lists



(a) Abelian setting (b) Non-abelian setting

The four families of quadrirational Yang-Baxter maps in the abelian and in the non-abelian setting. The morphisms Φ, Ψ, are respectively defined by $\Phi : R \to (\phi^{-1} \times id)R(id \times \phi)$ and $\Psi : R \to (\psi^{-1} \times id)R(id \times \psi)$, where ϕ, ψ symmetries.

The non-Abelian $\mathcal{F}-$list

The non-Abelian $\mathcal{F}-$list of quadrirational Yang-Baxter maps reads:

$$R : (u, p, v, q) \mapsto (x, p, y, q)$$

where:

$$\begin{aligned} x =& p \left(p(u-v)(1-v)^{-1} - (qu-pv)(q-v)^{-1} \right)^{-1} \\ & \qquad \left((u-v)(1-v)^{-1} - (qu-pv)(q-v)^{-1} \right), \\ y =& q\left((1-q)u + q - p + (p-1)v\right)\left((1-p)u + (p-q)uv + p(q-1)v\right)^{-1} u, \end{aligned} \qquad (\mathcal{F}_I \equiv \mathcal{F}_{1,1,1})$$

$$\begin{aligned} x =& q^{-1}(1-v)(u-v)^{-1}(qu-pv+p-q)v(1-v)^{-1}, \\ y =& p^{-1}(qu-pv+p-q)(u-v)^{-1}u, \end{aligned} \qquad (\mathcal{F}_{II} \equiv \mathcal{F}_{0,1,1})$$

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$$\begin{aligned} x =& q^{-1}v(u-v)^{-1}(qu-pv), \\ y =& p^{-1}(qu-pv)(u-v)^{-1}u, \end{aligned} \qquad (\mathcal{F}_{III} \equiv \mathcal{F}_{0,0,1})$$

$$\begin{aligned} x =& (u-v)^{-1}(u-v+p-q)v, \\ y =& (u-v+p-q)(u-v)^{-1}u, \end{aligned} \qquad (\mathcal{F}_{IV})$$

$$\begin{aligned} x =& v + (p-q)(u-v)^{-1}, \\ y =& u + (p-q)(u-v)^{-1}, \end{aligned} \qquad (\mathcal{F}_{V})$$

- **Can we do more? What about hierarchies of hierarchies?**

[9] P.K. 2022

- **Can we do more? What about hierarchies of hierarchies?**

- **Non-Abelian hierarchies of integrable difference systems**[9]

[9] P.K. 2022

What is known on hierarchies of maps I?[10]

$$(\nabla^k)_{ij} := \begin{cases} 0, & i \leq j \\ \delta_{i,j+k}, & i > j \end{cases} \qquad (\Delta^k)_{ij} := \begin{cases} \delta_{i+N-k,j}, & i < j \\ 0, & i \geq j \end{cases}$$

$$L^{(N,0)}(\mathbf{x};\lambda) := \mathbf{X} + \nabla^1 + \lambda\Delta^1\mathbf{X} = \begin{pmatrix} x^N & 0 & \cdots & 0 & \lambda \\ 1 & x^1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & x^{N-2} & 0 \\ 0 & 0 & & 1 & x^{N-1} \end{pmatrix}.$$

$$\mathcal{D} : (x^1, \ldots, x^N, y^1, \ldots, y^N) \mapsto (u^1, \ldots, u^N, v^1, \ldots, v^N),$$

$$u^i = (x^{i-1} - y^{i-1})x^i(x^i - y^i)^{-1}, \quad v^i = (y^{i-1} - x^{i-1})y^i(y^i - x^i)^{-1},$$

$i = 1, 2, \ldots, N.$

[10] Nijhoff, Papageorgiou, Capel and Quispel 1992, Doliwa 2013, Nimmo 2006, Kajiwara, Noumi and Yamada 2002

What is known on hierarchies of maps II?[11]

$$L^{(N,1)}(\mathbf{x};\lambda) := I_N + \nabla^1\mathbf{X} + \lambda\Delta^1\mathbf{X} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda\, x^N \\ x^1 & 1 & 0 & \cdots & 0 \\ 0 & x^2 & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & & x^{N-1} & 1 \end{pmatrix},$$

$$\mathcal{G} : (x^1,\ldots,x^N,y^1,\ldots,y^N) \mapsto (u^1,\ldots,u^N,v^1,\ldots,v^N),$$

$$u^i = (x^i - y^i)x^{i-1}(x^{i-1} - y^{i-1})^{-1}, \quad v^i = (y^i - x^i)y^{i-1}(y^{i-1} - x^{i-1})^{-1},$$

$i = 1, 2, \ldots, N.$

[11] Nijhoff, Papageorgiou, Capel and Quispel 1992, P.K 2021

Diagonal deformations[12]

Multiply $L^{(N,j)}$, $j = 0, 1$, from the left with

$$(P^{(N,0)}(\mathbf{x}))_{i,i} := \left(\alpha^i - x^{i-1}\right)^{-1}, \quad (P^{(N,1)}(\mathbf{x}))_{i,i} := \left(1 - \beta^i x^{i-1}\right)^{-1}, \quad \alpha^i, \beta^i \in C(\mathbb{D}),$$

obtain $\widehat{L}^{(N,j)}$, $j = 0, 1$. Specifically, there is

$$\widehat{L}^{(N,j)}(\mathbf{x}) := P^{(N,j)}(\mathbf{x}) L^{(N,j)}(\mathbf{x}), \qquad j = 0, 1.$$

Set $\alpha^i = \beta^i = 1$, so $P^{(N,0)} = P^{(N,1)} = P^{(N)}$

$$P^{(N)}(\mathbf{x}) := \begin{pmatrix} (1-x^N)^{-1} & 0 & \cdots & 0 & 0 \\ 0 & (1-x^1)^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & (1-x^{N-2})^{-1} & 0 \\ 0 & 0 & & 0 & (1-x^{N-1})^{-1} \end{pmatrix}$$

[12] P.K. 2022

The hierarchies $\mathcal{K}^{(j)}$, $j = 0, 1$

The hierarchies of maps

$$\mathcal{K}^{(0)} : (x^1, \dots, x^N, y^1, \dots, y^N) \mapsto (u^1, \dots, u^N, v^1, \dots, v^N),$$

where

$$\begin{aligned} u^i &= (1 - y^{i-1})^{-1}(x^{i-1} - y^{i-1})x^i(x^i - y^i)^{-1}(1 - y^i), \\ v^i &= (1 - x^{i-1})^{-1}(y^{i-1} - x^{i-1})y^i(y^i - x^i)^{-1}(1 - x^i), \end{aligned} \qquad i = 1, 2, \dots, N,$$

and

$$\mathcal{K}^{(1)} : (x^1, \dots, x^N, y^1, \dots, y^N) \mapsto (u^1, \dots, u^N, v^1, \dots, v^N),$$

where

$$\begin{aligned} u^i &= (1 - y^i)^{-1}(x^i - y^i)x^{i-1}(x^{i-1} - y^{i-1})^{-1}(1 - y^{i-1}), \\ v^i &= (1 - x^i)^{-1}(y^i - x^i)y^{i-1}(y^{i-1} - x^{i-1})^{-1}(1 - x^{i-1}) \end{aligned} \qquad i = 1, 2, \dots, N,$$

Properties of the hierarchies $\mathcal{K}^{(j)}$, $j = 0, 1$

1. Admit strong Lax matrices
2. Are birational
3. Are multidimensional compatible
4. Admit two non-equivalent families of multiplicative potentials
5. In terms of the potentials we obtain two Bäcklund related integrable hierarchies in vertex variables with rombic symmetry

Furthermore, if we impose the centrality assumptions

$$x^N \cdots x^2 x^1 = p \in C(\mathbb{D}), \quad y^N \cdots y^2 y^1 = q \in C(\mathbb{D}),$$

1. quadrirationality
2. The companion hierarchies define a hierarchies of Yang-Baxter maps
3. The first multiplicative potential leads to lattice-modified Gel'fand-Dikki hierarchy
4. The second multiplicative potential leads to lattice-NQC[13] Gel'fand-Dikki hierarchy

[13] Nijhoff-Quispel-Capel or $(Q3)_0$

Hierarchies of Yang-Baxter maps

The hierarchy of maps

$$c\mathcal{K}^{(1)}:(x^1,\dots,x^N,v^1,\dots,v^N)\mapsto(u^1,\dots,u^N,y^1,\dots,y^N),$$

where

$$\begin{aligned} u^i &= \left(p+v^iB^ig^{i-N+1}_{N-2}\right)\left(g^{i-N+1}_{N-1}\right)^{-1}, \\ y^i &= \left(f^{i+N-1}_{N-1}\right)^{-1}\left(q+f^{i+N-1}_{N-2}A^{i+1}x^i\right), \end{aligned} \qquad i=1,\dots,N,$$

1. is the companion of the hierarchy of maps $\mathcal{K}^{(1)}$;
2. is a hierarchy of Yang-Baxter maps.

The expressions g^i_{N-1},g^i_{N-2}, $i=1,\dots,N$, are determined by the recurrences

$$g^i_{n+2}=\left(v^{i+n}+B^{i+n+1}\right)g^i_{n+1}-v^{i+n}B^{i+n}g^i_n, \qquad n\in\mathbb{Z},$$

$$g^i_0=1, \qquad g^i_1=v^{i-1}+B^i, \qquad B^i:=\left(1-v^i\right)^{-1}x^{i-1}\left(1-v^{i-1}\right),$$

$i=1,\dots,N$. Similarly for the expressions f^i_{N-1},f^i_{N-2}, $i=1,\dots,N$.

The lattice-NQC Gel'fand-Dikki hierarchy

- $\mathcal{K}^{(0)}$, in terms of potentials ψ^i

$$\left(1-x^i\right)^{-1}x^i=\psi^i_1\left(\psi^i\right)^{-1},\quad \left(1-y^i\right)^{-1}y^i=\psi^i_2\left(\psi^i\right)^{-1}$$

reads:

$$\left(1+\psi^i_{m+1,n+1}\left(\psi^i_{m,n+1}\right)^{-1}\right)\left(1+\psi^{i-1}_{m,n+1}\left(\psi^{i-1}_{m,n}\right)^{-1}\right)$$
$$=\left(1+\psi^i_{m+1,n+1}\left(\psi^i_{m+1,n}\right)^{-1}\right)\left(1+\psi^{i-1}_{m+1,n}\left(\psi^{i-1}_{m,n}\right)^{-1}\right),$$

$$i=1,\dots N,\quad m,n\in\mathbb{Z}.$$

- Respects the rombic symmetry
- Add centrality and obtain lattice-NQC Gel'fand-Dikki hierarchy

$$\prod_{l=1}^{N}\left(1+\psi^{N-l+1}_{m,n}\left(\psi^{N-l}_{m+1,n}\right)^{-1}\right)^{-1}=p_m\in C(\mathbb{D}),$$
$$\prod_{l=1}^{N}\left(1+\psi^{N-l+1}_{m,n}\left(\psi^{N-l}_{m,n+1}\right)^{-1}\right)^{-1}=q_n\in C(\mathbb{D}),$$

- $N = 2$ and the $(Q3)_0$ integrable lattice equation

$$(\psi_{m,n+1} - p\psi_{m+1,n+1}) \left(\psi_{m,n+1} - \tfrac{q}{q^2-1}(q\psi_{m,n+1} - \psi)\right)^{-1}$$
$$= (\psi_{m+1,n} - q\psi_{m+1,n+1}) \left(\psi_{m+1,n} - \tfrac{p}{p^2-1}(p\psi_{m+1,n} - \psi)\right)^{-1}$$

- $N > 2$ and the $(Q3)_0$ hierarchy integrable lattice equation

$$\left(1 + \psi^1_{12}\left(\psi^1_2\right)^{-1}\right)\left(1 - q\prod_{l=1}^{N-1}\left(1 + \psi^l\left(\psi^l_2\right)^{-1}\right)\right)^{-1}$$
$$= \left(1 + \psi^1_{12}\left(\psi^1_1\right)^{-1}\right)\left(1 - p\prod_{l=1}^{N-1}\left(1 + \psi^l\left(\psi^l_1\right)^{-1}\right)\right)^{-1},$$

$$\left(1 + \psi^i_{12}\left(\psi^i_2\right)^{-1}\right)\left(1 + \psi^{i-1}_2\left(\psi^{i-1}\right)^{-1}\right) = \left(1 + \psi^i_{12}\left(\psi^i_1\right)^{-1}\right)\left(1 + \psi^{i-1}_1\left(\psi^{i-1}\right)^{-1}\right),$$

$$\left(1 - p\prod_{l=1}^{N-1}\left(1 + \psi^l_2\left(\psi^l_{12}\right)^{-1}\right)\right)^{-1}\left(1 + \psi^{N-1}_2\left(\psi^{N-1}\right)^{-1}\right)$$
$$= \left(1 - q\prod_{l=1}^{N-1}\left(1 + \psi^l_1\left(\psi^l_{12}\right)^{-1}\right)\right)^{-1}\left(1 + \psi^{N-1}_1\left(\psi^{N-1}\right)^{-1}\right),$$

$i = 2, \ldots, N-1.$

Lattice-potential pKdV as a bonus study

$$L(\mathbf{x};\lambda) := \begin{pmatrix} x^1 & (x^1+x^2)x^1-\lambda \\ 1 & x^1 \end{pmatrix},$$

- Introduce potentials

$$x^1 = \phi_1 - \phi, \qquad y^1 = \phi_2 - \phi,$$
$$\left(x^1+x^2\right)x^1 = \psi_1 + \psi, \quad \left(y^1+y^2\right)y^1 = \psi_2 + \psi.$$

- to obtain the rombic LpKdV

$$(\phi_{12}-\phi_2)(\phi_2-\phi) - (\phi_{12}-\phi_1)(\phi_1-\phi) = \psi_1 - \psi$$
$$(\psi_{12}+\psi_2)(\phi_2-\phi) + (\phi_{12}-\phi_2)(\psi_2+\psi) = (\psi_{12}+\psi_1)(\phi_1-\phi) + (\phi_{12}-\phi_1)(\psi_1+\psi$$

- Add centrality

$$(\phi_1-\phi)^2 - p = \psi_1 + \psi, \qquad (\phi_2-\phi)^2 - q = \psi_2 + \psi,$$

- Obtain D_4 LpKdV

$$(\phi_1-\phi_2)(\phi_{12}-\phi) + (\phi_{12}-\phi)(\phi_1-\phi_2) = 2(p-q).$$

Conclusions/What is next

- What about elliptic compatible maps?

- What about Liouville integrability of the (extended)transfer maps? What is the analogue of Poisson bracket in the non-Abelian setting?

СПАСИБО!