Applications of tropical algebra

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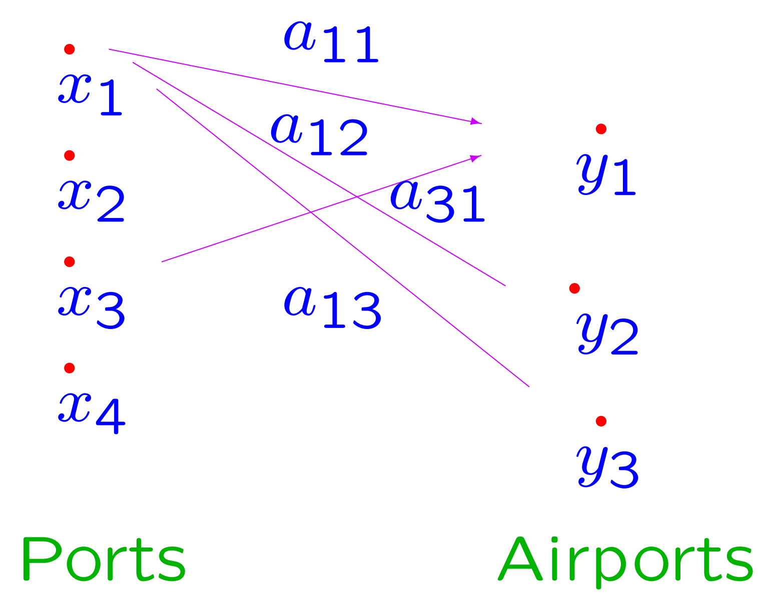
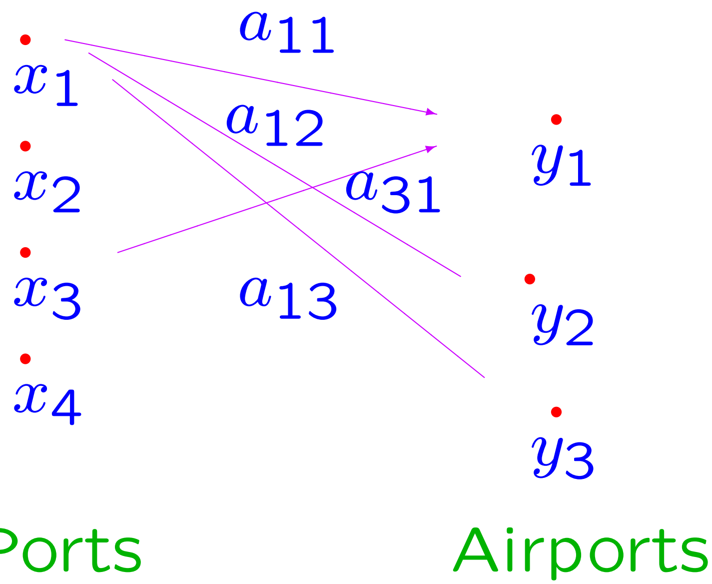
Problem 1. Scheduling problem. Tropical approach



$$\max\{x_1 + a_{11}, x_2 + a_{21}, \dots, x_k + a_{k1}\} = y_1$$

Scheduling problem. Tropical approach



$$\begin{cases} \max\{x_1 + a_{11}, x_2 + a_{21}, \ldots, x_k + a_{k1}\} = y_1 \\ \max\{x_1 + a_{12}, x_2 + a_{22}, \ldots, x_k + a_{k2}\} = y_2 \end{cases}$$

$$\left\{ \begin{array}{rcl} \max\{x_1 + a_{11}, x_2 + a_{21}, \ldots, x_k + a_{k1}\} & = & y_1 \\ \max\{x_1 + a_{12}, x_2 + a_{22}, \ldots, x_k + a_{k2}\} & = & y_2 \\ \max\{x_1 + a_{13}, x_2 + a_{23}, \ldots, x_k + a_{k3}\} & = & y_3 \end{array} \right.$$

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$$\max\{7, 2\} + \max\{5, 8\} = 15 \neq 12 = \max\{7 + 5, 2 + 8\}$$

$$\max \longleftrightarrow \oplus$$

$$+ \longleftrightarrow \otimes$$

$$\left\{\begin{array}{lcl} x_1 \otimes a_{11} \oplus x_2 \otimes a_{21} \oplus \ldots \oplus x_k \otimes a_{k1} & = & y_1 \\ x_1 \otimes a_{12} \oplus x_2 \otimes a_{22} \oplus \ldots \oplus x_k \otimes a_{k2} & = & y_2 \\ x_1 \otimes a_{13} \oplus x_2 \otimes a_{23} \oplus \ldots \oplus x_k \otimes a_{k3} & = & y_3 \\ x_1 \otimes a_{14} \oplus x_2 \otimes a_{24} \oplus \ldots \oplus x_k \otimes a_{k4} & = & y_4 \\ \ldots\ldots\ldots & & \ldots \\ x_1 \otimes a_{1s} \oplus x_2 \otimes a_{2s} \oplus \ldots \oplus x_k \otimes a_{ks} & = & y_s \end{array}\right.$$

$$A \otimes x = y$$

We have a system of tropical equations:

$$A \otimes x = b$$

How can we solve this?

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For our problem we need to find just one solution.

Definition. A semiring $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by ε);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by e);
- multiplication is distributive over addition on both sides;
- $s\varepsilon = \varepsilon s = \varepsilon$ for all $s \in \mathcal{S}$.

Examples:

- All rings are semirings
- $\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$

Are there semirings which are not "half of rings" ?

Dedekind, 1894

Let R be a ring. $\mathrm{Ideal}\,(R)$ be the set of its two-sided ideals, together with R.

$(\mathrm{Ideal}\,(R), +, \cdot)$ is a semiring:

addition and multiplication of ideals are defined elementwise

$\{0_R\}$, $\{R\}$ are neutral elements

$\mathrm{Ideal}\,(R)$ is not a ring since one can not subtract ideals

Now $\mathcal{S}$ is an antinegative semiring without zero divisors.

- Boolean algebras: algebras of subsets with respect to $\cup$ and $\cap$

Binary boolean algebra:

$$\begin{array}{ll} 0+0=0 & 0\cdot 0=0 \\ 1+0=1 & 1\cdot 0=0 \\ 0+1=1 & 0\cdot 1=0 \\ 1+1=1 & 1\cdot 1=1 \end{array}$$

- Max-algebras

Vorobiev, Cuninghame-Green, Olsder, Simon, Gondran, Minoux, ...

Definition. Max-algebra or tropical algebra:

$$\mathbb{R}_{\max} := (\mathbb{R}, \oplus, \otimes),$$

$$a \oplus b = \max\{a, b\}, \qquad a \otimes b = a + b,$$

$$\overline{\mathbb{R}_{\max}} := \mathbb{R}_{\max} \cup \{-\infty\}$$

zero is $-\infty$ and unit is 0.

$$2 \oplus 3 = 3; \;\; 2 \otimes 3 = 5$$

$$2^{\otimes^3} = ?$$

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$$\sqrt{-1} = -0.5$$

Equations

$$x \oplus 5 = 5 \iff \max\{5, x\} = 5 \iff x \in [-\infty;\ 5]$$

$$x \oplus 3 = 5 \iff \max\{3, x\} = 5 \iff x = 5$$

$$x \oplus 5 = 3 \iff \max\{5, x\} = 3 \iff x \in \emptyset$$

Matrices and vectors

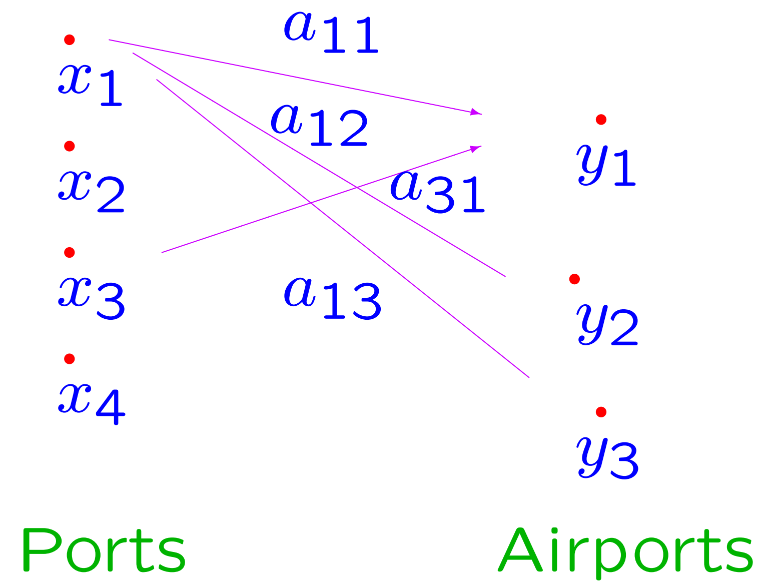
$$A = (a_{ij}), \;\; B = (b_{ij})$$

$$A \oplus B = (a_{ij} \oplus b_{ij}) = (\max\{a_{ij}, b_{ij}\})$$

$$[A \otimes B]_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k \{a_{ik} + b_{kj}\}$$

$$\alpha \otimes A = (\alpha \otimes a_{ij}) = (\alpha + a_{ij})$$

Scheduling problem. Tropical approach



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Identity matrix: $I = \text{diag}\,(0, \ldots, 0) = \begin{pmatrix} 0 & -\infty & \ldots & -\infty \\ -\infty & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\infty \\ -\infty & \ldots & -\infty & 0 \end{pmatrix}$

$$I \otimes A = A \otimes I = A$$

Definition Adjoint matrix to A is $A^{\bullet} = (a_{ij}^{\bullet})$, where $a_{ij}^{\bullet} = a_{ji}^{\otimes -1} = -a_{ji}$.

Lemma $A \otimes A^{\bullet} \geq I$.

Proof $(A \otimes A^{\bullet})_{ij} = \max_k \{a_{ik} - a_{kj}\} \geq -\infty$.

If $i = j$ then $(A \otimes A^{\bullet})_{ii} = \max_k \{a_{ik} - a_{ki}\} \geq 0$, since if $k = i$ then 0 is in the set. $\square$

Dual operations

$$
\begin{aligned}
a \oplus' b &= \min\{a, b\} \\
a \otimes' b &= a + b = a \otimes b
\end{aligned}
$$

For vectors and matrices:

$$
\begin{aligned}
A \oplus' B &= \left(a_{ij} \oplus' b_{ij} \right) \\
A \otimes' B &= \left(\bigoplus_k{}' a_{ik} \otimes' b_{kj} \right) = \left(\min_k (a_{ik} + b_{kj}) \right)
\end{aligned}
$$

Properties are the same

Notation: $\overline{x} = A^{\bullet} \otimes' b$

Theorem. x is a solution of $A \otimes x \leq b$ iff $x \leq \overline{x}$.

Proof.

$$A \otimes x \leq b \Leftrightarrow \max_j (a_{ij} + x_j) \leq b_i \ \forall\ i$$

$$\Leftrightarrow a_{ij} + x_j \leq b_i \ \forall\ i, j$$

$$\Leftrightarrow x_j \leq a_{ij}^{\otimes^{-1}} \otimes b_i = -a_{ij} + b_i \ \forall\ i, j$$

multiplicative inverses exist

$$\Leftrightarrow x_j \leq \min_i (-a_{ij} + b_i) \ \forall\ j$$

$$\Leftrightarrow x_j \leq \min_i (a_{ji}^{\bullet} + b_i) \ \forall\ j \quad \text{def. of } a_{ji}^{\bullet}$$

$$\Leftrightarrow x \leq A^{\bullet} \otimes' b = \overline{x}$$

Corollaries. Let $\overline{x} = A^{\bullet} \otimes' b$.

1. $A \otimes \overline{x} \leq b$.

2. Any solution x of $A \otimes x = b$ satisfies $x \leq \overline{x}$.

3. $A \otimes x = b$ has a solution iff $\overline{x}$ is a solution.

Some properties, we need

$A \leq B$ $(\Rightarrow)$ $A \otimes C \leq B \otimes C$ and $C \otimes A \leq C \otimes B$

The converse does not hold. Consider, $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}$.

Then $A \not\leq B$, but $A \otimes C = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} = B \otimes C$, i.e., $A \otimes C \leq B \otimes C$.

($\Leftarrow$): Ev. If $\overline{x}$ is a solution, then it exists.

($\Rightarrow$): Converse: let $\overline{x} = A^{\bullet} \otimes' b$ is not a solution.

1. $A \otimes \overline{x} \neq b$.

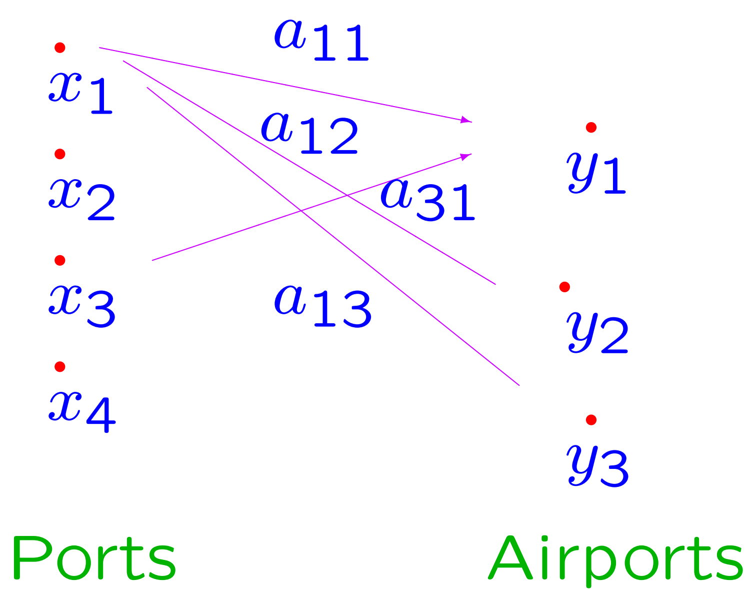
2. By Theorem for any $y \leq \overline{x}$ it holds that $A \otimes y \leq b$.

3. Consider $y = \overline{x}$. Then $A \otimes \overline{x} \leq b$, $A \otimes \overline{x} \neq b$. Hence, $A \otimes \overline{x} < b$.

4. For any $x < \overline{x}$ by monotonicity $A \otimes x < A \otimes \overline{x} < b$. Hence x is not a solution.

5. For any $x > \overline{x}$ by Corollary 2 ($A \otimes x = b \Rightarrow x \leq \overline{x}$) x is not a solution.

Thus there are NO solutions.



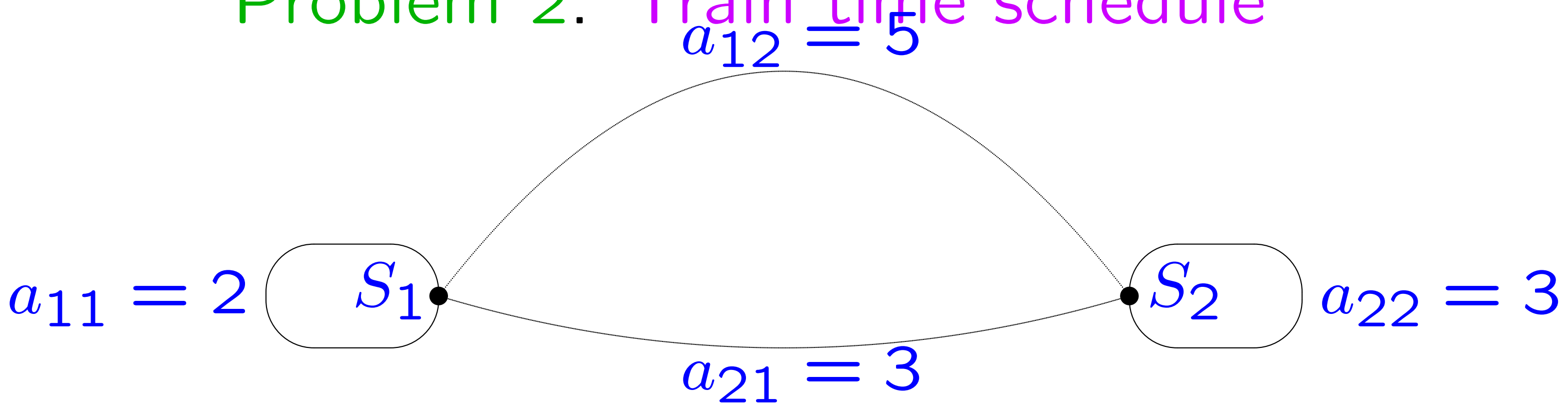
a_{11}
x_1
a_{12}
y_1
x_2
a_{31}
x_3
a_{13}
y_2
x_4
y_3
Ports
Airports

Example. $A = \begin{pmatrix} 2 & 1 & 3 & 3 \\ 4 & 3 & 1 & 1 \\ 0 & 2 & -\infty & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}.$

$$A^{\bullet} = \begin{pmatrix} -2 & -4 & 0 \\ -1 & -3 & -2 \\ -3 & -1 & \infty \\ -3 & -1 & -1 \end{pmatrix}$$

$$\overline{x} = A^{\bullet} \otimes' y = \begin{pmatrix} -2 & -4 & 0 \\ -1 & -3 & -2 \\ -3 & -1 & \infty \\ -3 & -1 & -1 \end{pmatrix} \otimes' \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

Problem 2. Train time schedule



1. Frequency must be maximal possible.

2. Frequency must be constant on all roads (regular time schedule).

3. Passengers must have a possibility to change a train at the station.

4. Trains must stay at stations the minimal possible intervals.

$A = (a_{ij}) = \left(\begin{smallmatrix} 2 & 5 \\ 3 & 3 \end{smallmatrix}\right)$. Thus

$$\begin{cases} x_1(k+1) & \geq & \max\{x_1(k)+2, x_2(k)+5\} \\ x_2(k+1) & \geq & \max\{x_1(k)+3, x_2(k)+3\} \end{cases}$$

By optimality

$$\begin{cases} x_1(k+1) & = & \max\{x_1(k)+2, x_2(k)+5\} \\ x_2(k+1) & = & \max\{x_1(k)+3, x_2(k)+3\} \end{cases}$$

If $x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$ is given then the rest is uniquely defined!

$$\begin{cases} x_1(k+1) = \max\{x_1(k)+2, x_2(k)+5\} \\ x_2(k+1) = \max\{x_1(k)+3, x_2(k)+3\} \end{cases}$$

$x_1(0) = x_2(0) = 0 \Rightarrow$

$x(k)$: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 11 \end{pmatrix}, \begin{pmatrix} 16 \\ 16 \end{pmatrix}, \ldots$

$x_1(0) = 1,\ x_2(0) = 0 \Rightarrow$

$x(k)$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 12 \end{pmatrix}, \begin{pmatrix} 17 \\ 16 \end{pmatrix}, \ldots$

The second schedule is more regular, isn't it ?

Is it possible to construct a better schedule?

Cycle $S_1 S_2$ takes 8 hours.

Hence, an interval between trains can not be less than

$$8/2 = 4$$

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Cycle S_1S_2 takes 8 hours.

Hence, an interval between trains can not be less than

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So, this is the optimal schedule!

$$\begin{cases} x_1(k+1) &= \max\{x_1(k)+2, x_2(k)+5\} \\ x_2(k+1) &= \max\{x_1(k)+3, x_2(k)+3\} \end{cases}, \qquad A = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}$$

$$\begin{matrix} \max \longrightarrow \oplus \\ + \longrightarrow \otimes \end{matrix} \quad : \quad x(k+1) = A \otimes x(k)$$

$$x(1) = A \otimes x(0)$$

$$x(2) = A \otimes x(1) = A \otimes (A \otimes x(0)) = A^{\otimes^2} \otimes x(0)$$

Similarly, $x(k) = A^{\otimes^k} \otimes x(0)$.

$x_1(0) = x_2(0) = 0 \Rightarrow x(k)$: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 11 \end{pmatrix}, \begin{pmatrix} 16 \\ 16 \end{pmatrix}, \ldots$

$x_1(0) = 1,\ x_2(0) = 0 \Rightarrow x(k)$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 12 \end{pmatrix}, \begin{pmatrix} 17 \\ 16 \end{pmatrix}, \ldots$

$A \in M_n$. Let $A \otimes v = \lambda \otimes v$, $v \neq -\infty$.

$$\lambda \otimes v := (\lambda \otimes v_i)_{i=1,\ldots,n} = (\lambda + v_i)_{i=1,\ldots,n}$$

v is an eigenvector, λ is an eigenvalue

If $x(0)$ is an eigenvector with an eigenvalue λ, then $x(k) = \lambda^{\otimes^k} \otimes x(0) (= k\lambda + x(0))$.

Initial conditions – eigenvectors $\Rightarrow$ regular timetable

Graphs and matrices

Directed graph G is (V, E), V − vertices,

$E \subseteq V \times V$ − edges

G is weighted if $w : D \to \mathbb{R}$ is defined

Path from i to j is $p = ((i_k, j_k) \in D(A), k = \overline{1, m})$ if

$$i = i_1, j_k = i_{k+1}, j_m = j$$

Length $|p|_l = m$, weight $|p|_w = \sum_{k=1}^{m} a_{i_{k+1} i_k}$.

A circuit is a closed path: $i = j$. It is elementary if $i_k \neq i_l \ \forall \ k, l$.

Def. If for any $i, j \in N \ \exists$ path from i to j (iRj), then G is strongly connected.

Def. A is indecomposable if $G(A)$ is strongly connected. Otherwise A is decomposable.

Theorem. A is indecomposable $\Rightarrow$ Eigenvalue

$$\lambda = \max_{\gamma} \frac{|\gamma|_w}{|\gamma|_l}$$

where γ – elem. circuit in $G(A)$.

Theorem. A is indecomposable

$$\Rightarrow \lambda = \min_{i=1,\ldots,n} \left(\max_{k=1,\ldots,n-1} \frac{(A^{\otimes^n})_{i,j} - (A^{\otimes^k})_{i,j}}{n-k} \right) \; \forall\, j$$

$$A^{+} = \bigoplus_{k=1}^{\infty} A^{\otimes^k}$$

Theorem. A is indecomposable $\Rightarrow$ Eigenvectors are weakly linear independent system of i'th columns of A_λ^{+}, where vertex i lies in the elementary circuit with the maximal average weight, $A_\lambda = A - \lambda J$, λ is the eigenvalue, $J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$.

A few words about linear independence over tropical semirings

Definition A system of elements, $\mathbf{m}_1, \ldots, \mathbf{m}_k$ in a semimodule M over a semiring $\mathcal{S}$ is *linearly dependent in the Gondran-Minoux sense* if there exist two subsets $I, J \subseteq K := \{1, \ldots, k\}$, $I \cap J = \emptyset, \quad I \cup J = K$ and scalars $\alpha_1, \ldots, \alpha_k \in \mathcal{S}$, $\neq (0, \ldots, 0)$, such that

$$\sum_{i \in I} \alpha_i \mathbf{m}_i = \sum_{j \in J} \alpha_j \mathbf{m}_j$$

Theorem. [Gondran, Minoux] Any $n + 1$ vectors of the size n are linearly dependent

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Minus: often there is no linear independent generating sets in a semimodule

Example [Butkovič, Cuninghame-Green]

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

are linearly dependent over $\mathbb{R}_{\max}$ since

$$\max\left\{\begin{bmatrix} 1-1 \\ 0-1 \\ -1-1 \end{bmatrix}, \begin{bmatrix} 3+0 \\ 0+0 \\ -3+0 \end{bmatrix}\right\} = \max\left\{\begin{bmatrix} 2+0 \\ 0+0 \\ -2+0 \end{bmatrix}, \begin{bmatrix} 4-1 \\ 0-1 \\ -4-1 \end{bmatrix}\right\}$$

$V = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$ contains no l.in. generating set. No 3 vectors generate V

$\forall$ 4 vectors from V are linearly dependent.

Are there better variants of Linear dependence?

Definition A subset P of elements in a semimodule M is called weakly linearly dependent if there is an element in P that can be expressed as a linear combination of other elements of P

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Plus: Any f.g. module has a finite weakly linearly independent generating set.

$\{x_1, \ldots, x_k\}$ are weakly linearly dependent.

$\Rightarrow x_i = \sum_{j \neq i} \lambda_j x_j$

$\Rightarrow \{x_j | j \neq i\}$ generates $\{x_1, \ldots, x_k\}$. Etc

Minus: There exist infinite systems of weakly linearly independent 3-vectors.

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[Butkovič, Cuninghame-Green]

Vectors $\begin{bmatrix} x_i \\ 0 \\ -x_i \end{bmatrix} \in \mathbb{R}^3_{\text{max}}$, $i = 1, 2, \ldots, m$ are weakly linearly independent for any m and for different x_i.

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Idea: due to max any linear combination disturbs either 0 in the middle or symmetry of x and $-x$.

Any 4 of these vectors are Gondran-Minoux linearly dependent!

weak linear dependence	$\Longrightarrow$	Gondran-Minoux linear dependence

$\not\Longleftarrow$:

Definition [Izhakian] A system of elements,

$$\mathbf{m}_1, \ldots, \mathbf{m}_k,$$

$\mathbf{m}_i = [m_i^1, \ldots, m_i^n]^t$, $i = 1, \ldots, k$, in a semimodule M is *strongly linearly dependent* if there exist two series of subsets $I_l, J_l \subseteq K := \{1, \ldots, k\}$, $I_l \cap J_l = \emptyset$, $I_l \cup J_l = K$, $l = 1, \ldots, n$, and $\alpha_1, \ldots, \alpha_k \in \mathcal{S}$, $\neq (0, \ldots, 0)$:

$$\sum_{i \in I_l} \alpha_i m_i^l = \sum_{j \in J_l} \alpha_j m_j^l$$

Gondran-Minoux linear dependence $\Longrightarrow$ strong linear dependence

$\not\Longleftarrow$:

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^3_{\text{max}}$$

are strongly linearly dependent (coefficients $0, 0, 0$, note that in max-algebra 0 is not a neutral element by addition, $-\infty$ is), but linearly independent by Gondran-Minoux.

$0 \otimes m = 0 + m = m$. Thus we have:

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Consider $I_1 = \{1, 2\}$, $J_1 = \{3\}$.

Then $x_1^1 \oplus x_2^1 = \max\{-1, 0\} = 0 = x_3^1$.

Similarly, for $I_2 = I_3 = \{1\}$, $J_2 = J_3 = \{2, 3\}$

$x_2^2 \oplus x_3^2 = \max\{-1, 0\} = 0 = x_1^2$ and

$x_2^3 \oplus x_3^3 = \max\{-1, 0\} = 0 = x_1^3$.