n-simplex equations and corresponding algebraic systems

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§ 1. Yang-Baxter equation and bi-groupoids

Let X be a set. A map

$$R : X \times X \to X \times X$$

is said to be a set-theoretic solution or simply solution for the Yang-Baxter equation (YBE):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} : X^3 \to X^3$ acts as R on the i-th and j-th factors and as identity map on the other factor.

Example

For arbitrary set X the map $P(x, y) = (y, x)$ gives a solution for the Yang-Baxter equation.

If a map $R : X \times X \to X \times X$ gives a solution for YBE, then the operator $S = PR : X \times X$ gives a solution for the Braid equation,

$$S_1 S_2 S_1 = S_2 S_1 S_2,$$

where $S_1 = S \times id$ and $S_2 = id \times S$ are operators on $X \times X \times X$.

This equation corresponds to relation in the braid group and the third Reidemeister move.

Writing $R(x,y) = (\sigma_y(x), \tau_x(y))$ for $x, y \in X$, we say that a solution (X, R) is

- bijective if R is a bijective map;
- non-degenerate if σ_x and τ_x are invertible for all $x \in X$;
- squarefree if $R(x,x) = (x,x)$ for all $x \in X$;
- involutive if $R^2 = id$.

A groupoid is a non-empty set with one binary algebraic operation.

A bi-groupoid is a non-empty set with two binary algebraic operations.

If (X, R), $R(x, y) = (\sigma_y(x), \tau_x(y))$, is a solution of the YBE, then we can define a bi-groupoid

$$(X; \cdot, *),$$

where

$$\cdot, * : X \times X \to X, \quad x \cdot y = \sigma_x(y), \quad y * x = \tau_y(x).$$

Proposition

Let $(X, \cdot, *)$ be a bi-groupoid and $R : X \times X \to X \times X$ given by $R(x, y) = (x \cdot y, y * x)$ for $x, y \in X$. Then the pair (X, R) is a solution of YBE if and only if the equalities

$$\begin{aligned}(x \cdot y) \cdot z &= (x \cdot (z * y)) \cdot (y \cdot z),\\ (y * x) \cdot (z * (x \cdot y)) &= (y \cdot z) * (x \cdot (z * y)),\\ (z * (x \cdot y)) * (y * x) &= (z * y) * x,\end{aligned}$$

hold for all $x, y, z \in X$.

Corollary

1. If $x \cdot y = y$ for all $x, y \in X$, then the pair (X, R) is a solution of the YBE if and only if the operation $*$ is right distributive, i.e.

$$(z * x) * (y * x) = (z * y) * x$$

for all $x, y, z \in X$.

2. If $y * x = y$ for all $x, y \in X$, then the pair (X, R) is a solution of the YBE if and only if the operation $\cdot$ is right distributive, i.e.

$$(z \cdot y) \cdot x = (z \cdot x) \cdot (y \cdot x)$$

for all $x, y, z \in X$.

If $\sigma_y = id$ for all $y \in X$ or $\tau_x = id$ for all $x \in X$, then the solution (X, R) is called by elementary solution.

Any elementary solution defines a groupoid structure on X.

If $R(x, y) = (\sigma_y(x), y)$ and $P(x, y) = (y, x)$, then

$$P\,R\,P(x, y) = (x, \sigma_x(y)).$$

A. Soloviev (2000) proves that any non-degenerate solution is conjugate to an elementary solution.

Proposition (A. Soloviev, 2000)

If $R(x,y) = (\sigma_y(x), \tau_x(y))$, $x, y \in X$, gives a non-degenerate solution for YBE on X, then it conjugates to a solution of the form:

$$R'(x,y) = (\sigma_x(\tau_{\sigma_y^{-1}(x)}(y)),\, y).$$

If for all $a, b \in X$ there exists a unique $x \in X$ such that

$$\tau_{\sigma_x^{-1}(a)}(x) = \sigma_a^{-1}(b),$$

then this solution is non-degenerate.

– Any elementary solution of the YBE defines a right distributive groupoid;

– any non-degenerate solution defines a rack;

– any non-degenerate squarefree solution define a quandle.

§ 2. Racks and quandles

A quandle is a groupoid which satisfies three axioms.

These axioms motivated by the three Reidemeister moves of diagrams of knots in the Euclidean space $\mathbb{R}^3$.

Quandles were introduced independently by S. Matveev and D. Joyce in 1982.

Definition

A rack is a non-empty set X with a binary algebraic operation

$$(a, b) \mapsto a * b$$

satisfying the following conditions:

(R1) For any $a, b \in X$ there is a unique $c \in X$ such that $a = c * b$;

(R2) Right distributivity: $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in X$.

A quandle X is a rack which satisfies the following condition:

(Q1) $a * a = a$ for all $a \in X$.

Let X be a rack. For any $a \in X$ define a map $S_a : X \longrightarrow X$, which sends any element $x \in X$ to element $x * a$, i. e. $xS_a = x * a$.

We shall call S_a the symmetry at a.

Proposition

For every $a \in X$ the symmetry S_a is an automorphism of X.

The simplest example of quandle is the so called trivial quandle.

A quandle X is called trivial if $a * b = a$ for all $a, b \in X$, i. e. any symmetry S_b is the trivial automorphism.

We see that a trivial quandle can contains arbitrary number of elements.

We shall denote the trivial quandle with n elements by T_n.

Example

If we define on the additive group $(\mathbb{Z}_n, +)$, $n > 1$, the operation $x * y \equiv x + 1 \pmod{n}$ for all $x, y \in \mathbb{Z}_n$, then $(\mathbb{Z}_n, *)$ is a rack which is not a quandle.

Many examples of quandles comes from groups.

Example

If G is a group and n is a natural number, then the set G equipped with the binary operations

$$a * b = b^{-n} a b^{n},$$

gives a quandle structure on G called the n-conjugation quandle, denoted by $Conj_n(G)$.

If $n = 1$, then we shall call this quandle the conjugation quandle and write $Conj(G)$.

If G is abelian group, then $Conj(G)$ is a trivial quandle.

Example

If G is a group, then the set G equipped with the binary operations

$$a * b = ba^{-1}b,$$

gives a quandle structure on G called the core quandle, denoted by $Core(G)$.

We have seen that the n-conjugation quandle and the core quandle are defined by the words

$$u(a,b) = b^{-n}ab^{n} \text{ and } v(a,b) = ba^{-1}b,$$

respectively, in arbitrary group G.

We can formulate

Question

Let $w = w(x, y)$ be a a reduced word in the free group $F_2 = F_2(x, y)$. Under what conditions for arbitrary group G the algebraic system $(G, *_w)$ with binary operation

$$g *_w h = w(g, h)$$

is a rack (quandle)?

If the algebraic system $(G, *_w)$ is a rack (respectively, quandle), then we call this rack (respectively, quandle) a verbal rack (respectively, verbal quandle) defined by the word w.

Theorem (V. B. – T. Nasybullov – M. Singh, 2019)

Let $w = w(x, y) \in F(x, y)$ be such that $Q = (G, *_w)$ is a rack for every group G. Then, in fact, Q is a quandle, and

$$w(x, y) = yx^{-1}y \text{ or } w(x, y) = y^{-n}xy^n \text{ for some } n \in \mathbb{Z}.$$

Example

If G is a group and $\varphi \in \mathrm{Aut}(G)$, then the set G with binary operation

$$a * b = \varphi(ab^{-1})b$$

gives a quandle structure on G, which we denote by $A_\varphi(G)$ and call the Generalised Alexander quandle.

Example

Let M be a module over the ring of Laurent polynomials $\mathbb{Z}\left[t, t^{-1}\right]$. Then the Alexander quandle $A_t(M)$ is the set M with the quandle operation given by

$$a * b = ta + (1-t)b.$$

If we put $t = -1$, then we get the Takasaki quandle.

§ 3. n-simplex equations: construction and known solutions

Suppose that we have 3 straight lines l_1, l_2, and l_3 on the plane $\mathbb{R}^2$.

The line l_1 intersects with l_2 in the point R_{12}, with the line l_3 in the point R_{13}, and the line l_2 intersects with l_3 in the point R_{23}.

We assume that all points R_{12}, R_{13}, and R_{23} are different and are vertices of a triangle (2-simplex).

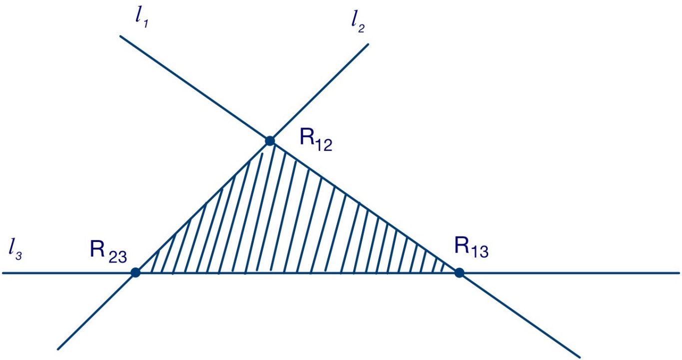

Figure: Geometric interpretation of YBE

Using the lexicographical order, we introduce the order on the vertices,

$$R_{12} < R_{13} < R_{23}.$$

Then the YBE is the equality of two words, where the first one is a word which we get if going around the vertices in the increasing order and the second word is a word which we get if going around the vertices in the decreasing order,

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12}.$$

To get the tetrahedron equation (3-SE) we increment the indices of all lines by 3 and get the triangle with the vertices R_{45}, R_{46}, and R_{56}.

Further, embed our plane $\mathbb{R}^2$ into a 3-space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, take a vertex R_{123}, which does not lie in $\mathbb{R}^2$.

Construct a straight line l_1, which connect R_{123} with the first vertex R_{45}; construct a straight line l_2, which connect R_{123} with the second vertex R_{46} and construct a straight line l_3, which connect R_{123} with the third vertex, R_{56}.

We construct a tetrahedron with the vertices R_{123}, R_{145}, R_{246}, and R_{356}.

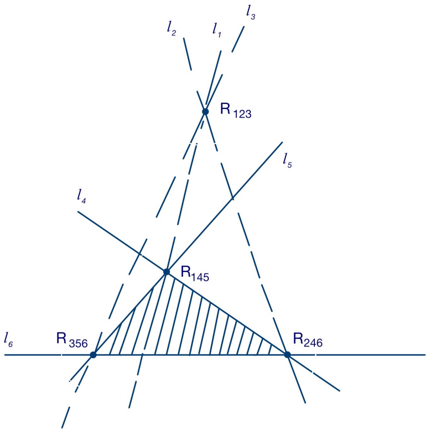

Figure: Geometric interpretation of TE

The TE or 3-SE is the equality of two words, where the first one is a word which we get if going around the vertices of the tetrahedron in the increasing order and the second word is a word which we get if going around the vertices in the decreasing order, i.e.

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}.$$

We have n-SE,

$$R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}} = R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}},$$

Define a shift

$$s_n : \mathbb{N} \to \mathbb{N}, \quad s_n(k) = k + (n+1),$$

and extend it to the multi-indexes by the rule, if

$$\overline{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$$

is a multi-index, then

$$s_n(\overline{k}) = (s_n(k_1), s_n(k_2), \ldots, s_n(k_n)) \in \mathbb{N}^n.$$

We get $(n+1)$-SE,

$$R_{1,2,\dots,n+1} R_{1,s_n(\overline{1})} R_{2,s_n(\overline{2})} \cdots R_{n+1,s_n(\overline{n})} =$$
$$= R_{n,s_n(\overline{n+1})} \cdots R_{2,s_n(\overline{2})} R_{1,s_n(\overline{1})} R_{1,2,\dots,n+1}.$$

Proposition

Let $R : X^n \to X^n$ be a solution of the n-SE.

1. If R is invertible, then its inverse R^{-1} is also a solution of the n-SE.
2. If $\sigma_1, \dots, \sigma_n$ are pairwise commuting endomorphisms of X, then a map $R : X^n \to X^n$ defined as

$$R(x_1, \dots, x_n) = (\sigma_1(x_1), \dots, \sigma_n(x_n)),$$

   is a solution of the n-SE.
3. If $\varphi \in Sym(X)$ is an arbitrary bijection of the set X onto itself, then $(\varphi)^{\times n} \circ R \circ (\varphi^{-1})^{\times n}$ is a solution of the n-SE.

§ 4. Rational solutions and Tropicalization

Let $\mathbb{R}(x_1, x_2, \ldots, x_n)$ be the field of rational fractions.

Any n-tuple $(r_1, r_2, \ldots, r_n)$ of rational fractions defines a map

$$R : \mathbb{R}^n \to \mathbb{R}^n$$

by the rule

$$R(x_1, x_2, \ldots, x_n) = (r_1, r_2, \ldots, r_n).$$

If $(\mathbb{R}, R)$ is a solution of n-SE, then it is called by a rational solution.

Let I_n be a subset on non-zero fractions $r = f/g \in \mathbb{R}(x_1, x_2, \ldots, x_n)$ such that

– all coefficients f are equal to 1 and the free term is equal to 0;

– g is equal to 1 or all its coefficients are equal to 1 and the free term is equal to 0.

Let PL_n be the set of piecewise linear functions $\mathbb{R}^n \to \mathbb{R}$.

Definition

A rational solution of n-SE,

$$R(x_1, x_2, \ldots, x_n) = (r_1, r_2, \ldots, r_n),$$

is said to be a I-rational solution if all r_i lie in I_n.

Example

It is easy to see that the famous electric solution of TE,

$$R_E(x,y,z)=\left(\frac{xy}{x+z+xyz},\ x+z+xyz,\ \frac{yz}{x+z+xyz}\right)$$

and the solution that is obtained from R_E by removing terms of degree three,

$$R_e(x,y,z)=\left(\frac{xy}{x+z},\ x+z,\ \frac{yz}{x+z}\right),$$

are I-rational solutions.

Definition

The tropicalization is a function $^t : I_n \to PL_n$ that is defined on $r = f/g \in I_n$, where

$$f = \sum_{i_1+\ldots+i_n>0} \alpha_{i_1\ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \quad g = \sum_{j_1+\ldots+j_n>0} \beta_{j_1\ldots j_n} x_1^{j_1} \ldots x_n^{j_n}$$

by the rule

$$r^t = \max_{i_1+\ldots+i_n>0} \{i_1 x_1 + \ldots + i_n x_n\} - \max_{j_1+\ldots+j_n>0} \{j_1 x_1 + \ldots + j_n x_n\}$$

for $g \neq 1$, and by the rule

$$r^t = \max_{i_1+\ldots+i_n>0} \{i_1 x_1 + \ldots + i_n x_n\}$$

for $g = 1$.

Note that r^t can be obtained from r using the following recursive procedure.

Proposition

Let r, r_1, r_2 be rational functions from I_n. Then

1. if $r = x_i$, then $r^t = x_i$, for $i = 1, \ldots, n$;
2. $(r_1 + r_2)^t = \max\{r_1^t,\ r_2^t\}$;
3. $(r_1 r_2)^t = r_1^t + r_2^t$;
4. $\left(\frac{r_1}{r_2}\right)^t = r_1^t - r_2^t$.

Definition

Let

$$R(x_1, \dots, x_n) = (r_1, r_2, , \dots, r_n) \in I_n^n$$

be a rational vector-valued map of n variables, where $r_1, r_2, \dots, r_n$ lie in I_n. Define the tropicalization of the rational map R componentwise:

$$R^t(x_1, \dots, x_n) := (r_1^t(x_1, \dots, x_n), \dots, r_n^t(x_1, \dots, x_n)).$$

Example

The tropicalization of R_E gives

$$R_E^t(x, y, z) = (x + y - M,\ M,\ y + z - M)\,,$$

where $M = \max\{x, z, x + y + z\}$.

The tropicalization of R_e gives

$$R_e^t(x, y, z) = (x + y - \max\{x, z\},\ \max\{x, z\},\ y + z - \max\{x, z\}).$$

One can check that R_E^t and R_e^t are solutions of TE.

The solution R_E^t consists from 3 linear pieces:

$$R_1(x, y, z) = (y,\ x,\ y + z - x),$$

$$R_2(x, y, z) = (x + y - z,\ z,\ y),$$

$$R_3(x, y, z) = (-z,\ x + y + z,\ -x).$$

One can check that $(\mathbb{R}, R_1)$ and $(\mathbb{R}, R_2)$ are solutions of TE, but $(\mathbb{R}, R_3)$ is not.

Question

Is there some general procedure to produce rational solutions from linear ones?

Theorem

If

$$(\mathbb{R}, R), \quad R \in I_n^n$$

is a I-rational solution of the n-simplex equation, then its tropicalization

$$(\mathbb{R}, R^t)$$

is a piecewise linear solution of the n-simplex equation.

§ 5. Group extensions and parametric Yang-Baxter equation

Let a group G be an extension of H by K,

$$1 \to H \xrightarrow{i} G \xrightarrow{i} K \to 1,$$

M. Preobrazhenskaya and D. Talalaev (2021) in the case of abelian K, define a quandle structure $Conj(G) = (G, *)$, $g * h = h^{-1}gh$, on G.

In this case the map

$$R(g, h) = (g, h * g),$$

gives a solution for YBE.

Using R they construct a solution of a 3-parametric YBE,

$$R_{12}^{a,b}\, R_{13}^{a,c}\, R_{23}^{b,c} = R_{23}^{b,c}\, R_{13}^{a,c}\, R_{12}^{a,b}, \quad a, b, c \in K,$$

on H.

We give a more general construction.

We will present elements of G as the set of pairs

$$(x, a) \in H \times K$$

and assume that $i(x) = (x, 1)$ and $j((x, a)) = a$.

Then the multiplication on G is defined by the rule

$$(x, a) \circ (y, b) = (x \underset{a,b}{\circ} y, a \circ b) \text{ for some } x \underset{a,b}{\circ} y \in H,$$

where $a \circ b$ is the multiplication on K.

For the multiplication on H we use the same symbol $\circ$. These agree with the formulas

$$(1, a) \circ (1, b) = (1, a \circ b), \quad (x, 1) \circ (y, 1) = (x \circ y, 1).$$

Suppose, that on G is defined a binary algebraic operation

$$* : G \times G \to G$$

such that

1. $(G, *)$ is a right distributive groupoid;
2. H is closed under multiplication $*$;
3. $*$ defines a right distributive groupoid on K.

On the set of pairs

$$(x, a) \in H \times K$$

define the multiplication

$$(x, a) * (y, b) = (x \underset{a,b}{*} y, a * b) \text{ for some } x \underset{a,b}{*} y \in H.$$

Hence, on H we have operation $*$ and a set of operations

$$\{ \underset{a,b}{*} \mid a, b \in K \}.$$

The map

$$R(g,h) = (g, h * g), \quad g, h \in G,$$

defines a solution of the YBE on G.

Proposition

If

$$R^{u,v}(x,y) = (x, y \underset{v,u}{*} x), \quad u, v \in K$$

is the parametric map $H \times H \to H \times H$, then for any $a, b, c \in K$ the following equality

$$R_{12}^{a,b}\, R_{13}^{a,c*b}\, R_{23}^{b,c} = R_{23}^{b*a,c*a}\, R_{13}^{a,c}\, R_{12}^{a,b}$$

holds in H.

Corollary

If $(K, *)$ is a trivial right distributive groupoid, i.e. $u * v = u$ for any $u, v \in K$, then for any $a, b, c \in K$ the following equality

$$R_{12}^{a,b} \, R_{13}^{a,c} \, R_{23}^{b,c} = R_{23}^{b,c} \, R_{13}^{a,c} \, R_{12}^{a,b}$$

holds in H.

§ 6. Tetrahedral equation and ternoids

An algebraic system with one ternary operation is called by ternar, an algebraic system with k ternary operations is called by k-ternoid.

If

$$R = (f, g, h) : X^3 \to X^3$$

is a solution of the TE on some set X, then we can define on X three ternar operations

$$[a, b, c] = f(a, b, c), \quad \langle a, b, c\rangle = g(a, b, c), \quad \{a, b, c\} = h(a, b, c), \quad a, b, c \in X.$$

Hence, a solution of TE defines a 3-ternoid.

Proposition

Let $(X, [\cdot,\cdot,\cdot], \langle\cdot,\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a 3-ternoid. Then it defines a solution of the TE if and only if the following equalities hold

$$[[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]] = [[x, y, z], t, p],$$

$$\langle [x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\rangle = [\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle$$

$$\{[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\} =$$

$$= [\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}],$$

$$\langle x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\rangle = \langle \langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\rangle,$$

$$\{x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\} = \langle \{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t,$$

$$\{y, t, \{z, p, q\}\}) = \{\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}\}.$$

for all $(x, y, z, t, p, q) \in X^6$.

Corollary 1

Let $(X, [\cdot, \cdot, \cdot])$ be a ternar. The map

$$R(a, b, c) = ([a, b, c], b, c), \quad a, b, c \in X.$$

gives a solution of TE if and only if

$$[[x, t, p], [y, t, q], [z, p, q]] = [[x, y, z], t, p], \quad \text{for all } x, y, z, t, p, q \in X.$$

Corollary 2

Let $(X, [\cdot,\cdot,\cdot])$ be a ternar. The map

$$R(a,b,c) = (a, [a,b,c], c), \quad a,b,c \in X.$$

gives a solution of the TE if and only if

$$[[x,y,z],[x,t,p],q] = [x,[y,t,q],[z,p,q]], \quad \text{for all } x,y,z,t,p,q \in X.$$

Corollary 3

Let $(X, [\cdot,\cdot,\cdot])$ be a ternar. The map

$$R(a,b,c) = (a, b, [a,b,c]), \quad a,b,c \in X.$$

gives a solution of the TE if and only if

$$[[x,y,z],[x,t,p],[y,t,q]] = [y,t,[z,p,q]], \text{ for all } x,y,z,t,p,q \in X.$$

Let

$$P_{13} : X^3 \to X^3, \quad P_{13}(x, y, z) = (z, y, x),$$

be the permutation of the first and the third components.

If R is an elementary solution of the first type, then

$$P_{13}\, R\, P_{13}$$

is an elementary solution of the third type.

Hence, we have to study elementary solutions of the first and the second types.

It is known that any n-ary function on a set can be expressed as composition of binary functions.

Question

Is it possible to present our ternary operations as composition of binary ones?

We will call a 4-groupoid $(X, *, \circ, \triangleleft, \triangleright)$ by IE-groupoid if it satisfies the axioms

1) $x \triangleright (y * z) = (x \triangleright y) * (x \triangleright z)$,

2) $(x \circ y) \triangleleft z = (x \triangleleft z) \circ (y \triangleleft z)$,

3) $(x * y) \circ (z * w) = (x \circ z) * (y \circ w)$,

4) $(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z)$,

5) $(x * y) \triangleleft z = x \triangleright (y \circ z)$,

for all $x, y, z, w \in X$.

Proposition

Any IE-groupoid $(X, *, \circ, \triangleleft, \triangleright)$ gives an elementary solution (X, R) of TE if we put

$$R(x, y, z) = (x,\ x \triangleright (y \circ z),\ z), \quad x, y, z \in X.$$

Example

Let V be a vector space, define 4-groupoid $(V, *, \circ, \triangleleft, \triangleright)$ with operations:

$$x * y := (1-\beta)x + \beta y,$$
$$x \circ y := \beta x + (1-\beta)y,$$
$$x \triangleleft y := (1-\beta)x + y,$$
$$x \triangleright y := x + (1-\beta)y,$$

where β is some endomorphism of the vector space V. Then this 4-groupoid is IE-groupoid and gives the solution

$$R(x, y, z) = (x, (1-\beta)x + \beta y + (1-\beta)z, z).$$

On the other side, suppose that (X, R) is an elementary solution of TE,

$$R(x, y, z) = (x, [x, y, z], z),$$

such that there is $c \in X$ for which $[c, c, c] = c$, and an unary operation $\{\cdot\} : X \to X$,

$$\{[c, x, c]\} = [c, \{x\}, c] = x, \quad [\{x\}, \{y\}, c] = \{[x, y, c]\},$$

$$[c, \{x\}, \{y\}] = \{[c, x, y]\}.$$

Proposition

If we put

$$x * y = [x, y, c], \qquad x \circ y = [c, x, y],$$

$$x \triangleright y = [x, \{y\}, c], \qquad x \triangleleft y = [c, \{x\}, y],$$

then we get an IE-groupoid.

Example

If (V, R) is a solution, where V is a vector space and

$$R(x, y, z) = (x, (1-\beta)x + \beta y + (1-\beta)z, z), \quad \beta \in Aut(V),$$

then by taking $c := 0$ and $\{x\} := \beta^{-1}x$ we get a IE-groupoid.

In the paper
S. Igonin, V. Kolesov, S. Konstantinou-Rizos, M. M. Preobrazhenskaia, Tetrahedron maps, Yang-Baxter maps, and partial linearisations, arXiv:2106.09130
were constructed affine solution of TE.

What IE-groupoids corresponds to elementary solutions?

§ 7. Verbal solutions for the tetrahedral equation

Let G be a group. A verbal solution (G, R) of the n-SE is a solution

$$R(g_1, \dots, g_n) = (w_1(g_1, \dots, g_n), w_2(g_1, \dots, g_n), \dots, w_n(g_1, \dots, g_n)),$$

where $w_i = w_i(x_1, \dots, x_n)$ are reduced words in the free group $F_n = \langle x_1, \dots, x_n \rangle$.

A verbal solution R of the n-SE is said to be l-elementary if it does not fix only l-th component.

If (G, R) is verbal a solution for the n-SE and $\varphi : G \to K$ is a homomorphism, then it induces a solution $(\varphi(G), R^\varphi)$, where

$$R^\varphi(\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_n)) = (\varphi(g_1'), \varphi(g_2'), \ldots, \varphi(g_n')),$$

if

$$R(g_1, g_2, \ldots, g_n) = (g_1', g_2', \ldots, g_n').$$

For arbitrary group G there is a map $R : G^2 \to G^2$ which is an elementary solution for the YBE.

For example, we can take any quandle on G ($Conj_n(G)$, or $Core(G)$) and construct elementary solution on G.

Question

Let F be a non-abelian free group. Is there a map $R : F^n \to F^n$, $n > 2$, that gives a bijective non-trivial (elementary) solution for n-SE?

By a trivial solution we mean a permutation of components or solution which comes from a solution of $(n-1)$-SE.

In the case $n = 3$ a description of verbal 3-elementary solutions gives

Theorem

Let $R : G^3 \to G^3$ be a verbal 3-elementary solution of TE for every group G, then it has one of the following forms:

1. $R(x, y, z) = (x, y, yx^{-1})$,
2. $R(x, y, z) = (x, y, x^{-1}y)$,
3. $R(x, y, z) = (x, y, w(y, z))$, where $R'(y, z) = (y, w(y, z))$ is a solution of YBE.

Definition

Let $R : F_n \to F_n$ be an endomorphism

$$R(x_1, \ldots, x_n) = (w_1(x_1, \ldots, x_n), \ldots, w_n(x_1, \ldots, x_n)),$$

$\mathcal{V}$ be a variety of groups. We will say that R defines a verbal universal $\mathcal{V}$-solution for n-SE, if (G, R) is a solution for any group $G \in \mathcal{V}$.

If $\mathcal{V}$ is variety of abelian groups, we say on verbal universal abelian solution, if $\mathcal{V}$ is variety of nilpotent groups, we say on verbal universal nilpotent solution, and so on.

Using a result of J. Hietarinta (1992) one can find all verbal universal abelian solutions of TE.

Proposition

Let G be an abelian group, $\alpha, \beta, \gamma \in \mathbb{Z}$. Then any verbal solutions to the TE either come from a solution of YBE or has one of the following forms:

$$(a, b, c) \mapsto (a^{-\alpha}, a, a^{\alpha} b c^{\beta});$$
$$(a, b, c) \mapsto (a^{\alpha}, a, a^{-\beta} b c^{\beta});$$
$$(a, b, c) \mapsto (b c^{-\alpha}, ac, c^{\alpha});$$
$$(a, b, c) \mapsto (b, a, a^{-\alpha} b c^{\alpha});$$
$$(a, b, c) \mapsto (a^{\alpha} b^{1-\alpha\beta} c^{\alpha(\beta\gamma-1)}, b^{\beta} c^{1-\beta\gamma}, c^{\gamma});$$
$$(a, b, c) \mapsto (a^{\alpha} b^{1-\alpha\beta} c^{\gamma(\alpha\beta-1)}, b^{\beta} c^{1-\beta\gamma}, c^{\gamma});$$
$$(a, b, c) \mapsto (a^{\alpha}, a^{1-\alpha\beta} b^{\beta} c^{1-\beta\gamma}, c^{\gamma});$$
$$(a, b, c) \mapsto (a^{\alpha} b^{1-\alpha\beta}, b^{\beta}, b^{1-\beta\gamma} c^{\gamma}),$$

§ 8. Representation of the braid group and parametric Yang-Baxter equation

The braid group B_n, $n \geq 2$, on n strands is generated by $(n-1)$ elements

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$$

and is defined by the relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i &= 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i-j| &\geq 2. \end{aligned}$$

These groups are very important in low dimensional topology. In particular, they are used for construction of link invariants and invariants of 3-manifolds.

A bijective solution (X, R) of the YBE gives a representation

$$\varphi_R : B_n \to Sym(X^n)$$

of the braid group B_n into the group of permutations of X^n by the formula

$$\varphi_R(\sigma_i) = id^{i-1} \times R \times id^{n-i-1}, \quad i = 1, 2, \ldots, n-1.$$

The virtual braid group VB_n, $n \geq 2$, on n strands is generated by $2(n-1)$ elements

$$\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \rho_2, \dots, \rho_{n-1},$$

and is defined by the relations of B_n, relations of the symmetric group $S_n = \langle \rho_1, \rho_2, \dots, \rho_{n-1} \rangle$, and the mixed relations

$$\begin{aligned} \sigma_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \sigma_{i+1}, & i &= 1, 2, \dots, n-2, \\ \rho_i \sigma_j &= \sigma_j \rho_i, & |i-j| &\geq 2. \end{aligned}$$

These groups were introduced by Kauffman (1996) for the studying virtual links which are some generalization of the classical links.

If we have a pair of bijective solutions of YBE (X, R) and (X, S) on the same set X such that S is involutive and R and S satisfy the equality

$$(R \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times R),$$

then we can define a representation

$$\varphi_{R,S} : VB_n \to Sym(X^n)$$

of the virtual braid group VB_n by the formulas

$$\varphi_{R,S}(\sigma_i) = id^{i-1} \times R \times id^{n-i-1}, \quad , i = 1, 2, \ldots, n-1,$$

$$\varphi_{R,S}(\rho_i) = id^{i-1} \times S \times id^{n-i-1}, \quad i = 1, 2, \ldots, n-1.$$

On the other side, it is possible to find solutions for the parametric YBE, using representations of B_n or VB_n.

We will use a representation which was suggested by S. Kamada,

$$\varphi : VB_n \longrightarrow Aut(F_{n,3n}).$$

Let

$$F_n = \langle x_1 \dots x_n \rangle$$

be a free group,

$$\mathbb{Z}^{3n} = \langle w_1, \dots, w_n, u_1, \dots, u_n, v_1, \dots, v_n \rangle$$

be a free abelian group,

$$F_{n,3n} = F_n * \mathbb{Z}^{3n}$$

is their free product.

This representation is defined by the action on the generators:

$$\varphi(\sigma_i) : \begin{cases} x_i \longrightarrow x_i x_{i+1}^{u_i} x_i^{-w_{i+1}u_{i+1}}, \\ x_{i+1} \longrightarrow x_i^{w_{i+1}}, \end{cases} \qquad \varphi(\sigma_i) : \begin{cases} w_i \longrightarrow w_{i+1}, \\ w_{i+1} \longrightarrow w_i, \end{cases}$$

$$\varphi(\sigma_i) : \begin{cases} u_i \longrightarrow u_{i+1}, \\ u_{i+1} \longrightarrow u_i, \end{cases} \qquad \varphi(\sigma_i) : \begin{cases} v_i \longrightarrow v_{i+1}, \\ v_{i+1} \longrightarrow v_i, \end{cases}$$

$$\varphi(\rho_i) : \begin{cases} x_i \longrightarrow x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longrightarrow x_i^{v_{i+1}}, \end{cases} \qquad \varphi(\rho_i) : \begin{cases} w_i \longrightarrow w_{i+1}, \\ w_{i+1} \longrightarrow w_i, \end{cases}$$

$$\varphi(\rho_i) : \begin{cases} u_i \longrightarrow u_{i+1}, \\ u_{i+1} \longrightarrow u_i, \end{cases} \qquad \varphi(\rho_i) : \begin{cases} v_i \longrightarrow v_{i+1}, \\ v_{i+1} \longrightarrow v_i. \end{cases}$$

Theorem

Let $G = B * A$ be the free product of arbitrary group B and an abelian group A. Then the maps

$$R_{u,v,w}(x, y) = (x^w, xy^u x^{-wv}),$$

give a solution for the 5-parametric Yang-Baxter equation:

$$R^{12}_{u,v,w} R^{13}_{u,p,q} R^{23}_{v,p,q} = R^{23}_{v,p,q} R^{13}_{u,p,q} R^{12}_{u,v,w}$$

for $u, v, w, p, q, u_1, v_1, u_2, v_2 \in A$.

Theorem

Let $G = B * A$ be the free product of arbitrary group B and an abelian group A. Then the map

$$T_{u,v}(x, y) = (x^u, y^v), \quad u, v \in A,$$

gives a solution for the 6-parametric Yang-Baxter equation:

$$T^{12}_{u,v} T^{13}_{u_1,v_1} T^{23}_{u_2,v_2} = T^{23}_{u_2,v_2} T^{13}_{u_1,v_1} T^{12}_{u,v}, \quad u, u_1, u_2, v, v_1, v_2 \in A.$$

Problems

1. Find verbal universal solutions for n-simplex equations in class of all groups.
2. Find connections between group extensions and quandle extensions.
3. We know that braid group corresponds to YBE. What group corresponds to n-simplex equations?
4. With the YBE are connected braided categories. What categories correspond to n-simplex equations?

Thank you!