Geometry

Poisson manifold : $M, \{\ ,\ \}$

Complete system of functions in involution

examples and counterexamples

$\psi_1, \dots \psi_m, \quad \{\psi_i, \psi_j\} = 0$

$m = \dim M - \frac{1}{2} rk\ (\{,\})$

Bi – Poisson manifold : $M, \{\ ,\ \}_1, \{\ ,\ \}_2$

Complete system of functions in bi – involution

$\psi_1, \dots \psi_m, \quad \{\psi_i, \psi_j\}_1 = \{\psi_i, \psi_j\}_2 = 0$

$m = \dim M - \frac{1}{2} max\ (rk\ \{,\}_1 + \lambda\{,\}_2)$

$\left\{\begin{matrix} \{\ ,\ \}_1 - linear \\ \{\ ,\ \}_2 - const \end{matrix}\right\} \Rightarrow M = g^*, \{\ \} - \text{Lie Poisson}, \{\ \}_A - \text{"frozen"}, A \in g^*$

symplectic leaves = orbits of coadjoint representation

$m = \dim g - \frac{1}{2} max\ (\dim orbits) = \frac{1}{2}(\dim g + ind\ g)$

g – classical Lie algebra

$\{\ \}$ – *Lie Poisson*, $\{\ \}_A$ – *frozen*, $A \in g$

$g \simeq g^*$ *via invariant scalar product*

$\{\phi, \psi\}(X) = (X, [d_x\phi, d_x\psi])$

$\{\phi, \psi\}_A(X) = (A, [d_x\phi, d_x\psi])$

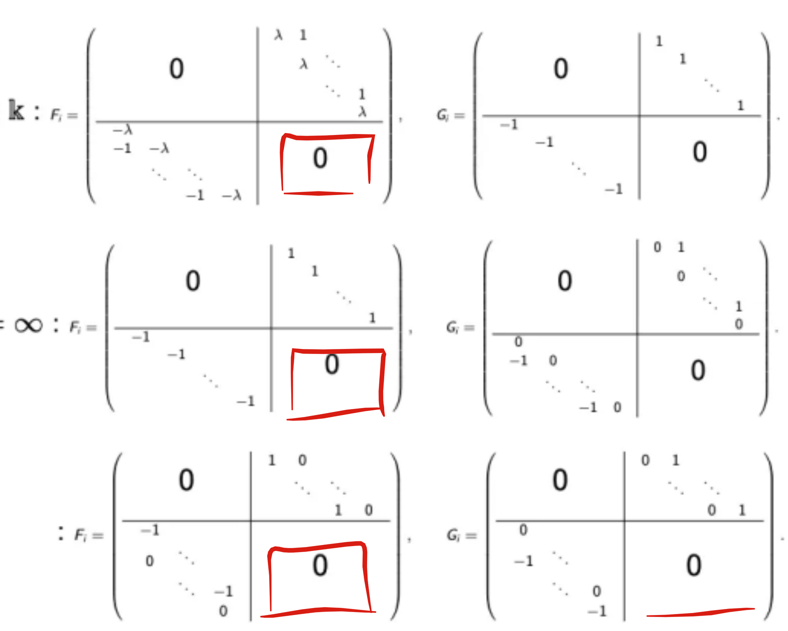
$\{\psi_i\}$ *form complete system of functions in biinvolution*

$\left\{\begin{matrix} \{\psi_i, \psi_j\} = \{\psi_i, \psi_j\}_A = 0 \\ m = \text{maximum possible} \end{matrix}\right\}$

Jordan, $\lambda \in \mathbb{k}$:

Jordan, $\lambda = \infty$:

Kronecker,

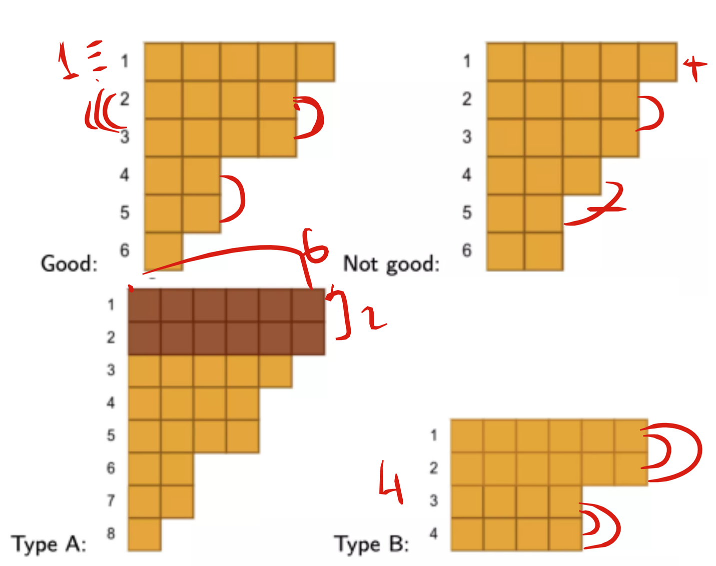


A – regular

functions = $\{coeffs\ over\ t\ of\ f_i(X - tA), \forall i\}$

Thus, we obtained shift argument method

A – singular



con

nilp

Results

Kronecker	Jordan
$g = gl_n, sl_n, sp_{2n}$	
for all A	for all A
$g = so_{2n+1}$	
good A correctable A	good A
$g = so_{2n}$	
good A correctable A	good semisimple A

Sheets

Thm. For any matrix $A \in g$, *where* $g = gl_n, sl_n$ *or* sp_{2n}, *the Kronecker indices of the pair* (B, B_A) *are constant within a sheet. For* $g = so_{2n+1}$ *and* so_{2n} *that's not true.*

Construction

$\boxed{dc_k = -C_k(X - tA)}$

Thm. For nilpotent A coeffs of $\{c_k, k = 0 \dots n-1\}$ *form a complete system of functions in biinvolution*

$\boxed{dr_k = -Q_k(X - tA)}$

Thm. For any A coeffs s of $\{r_k, k = 0 \dots n-1\}$ *form a complete system of functions in biinvolution*

Algebra

g: gl.

B, B_A – *skew – symmetric bilinear forms on* $V = g \otimes \mathbb{K}$ *over* $\mathbb{K} = \mathbb{C}(g)$

$B(d\phi, d\psi), \quad B(u,v) = (X, [u,v])$

$B_A(d\phi, d\psi), \quad B_A(u,v) = (A, [u,v])$

$\{d\psi_i\}$ *form a basis of bi – Lagrangian subspace w.r.t.* (B, B_A)

$\left\{\begin{matrix} B(d\psi_i, d\psi_j) = B_A(d\psi_i, d\psi_j) = 0 \\ m = \text{maximum possible} \end{matrix}\right\}$

bring (B, B_A) *to Jordan – Kronecker canonical form*

Kronecker part

invariants $\quad m_0, \dots, m_{n-1}, n = ind\ g$

Kronecker Method :

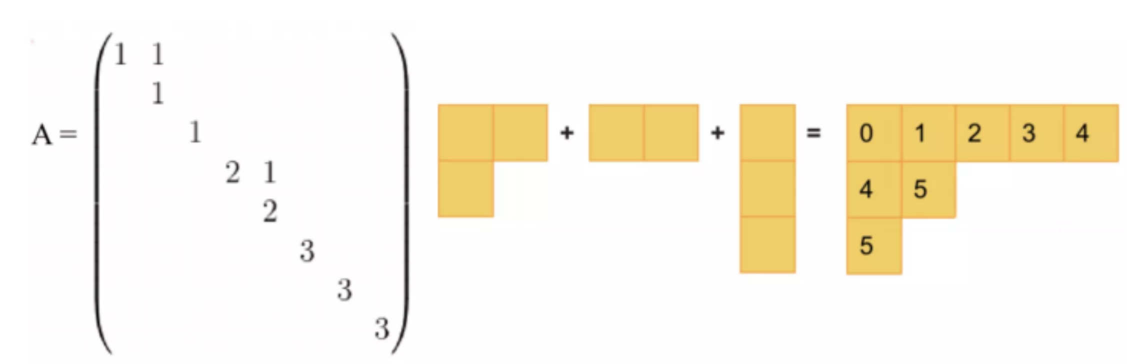
1. *Consider* $Z = Ker(B - tB_A)$ *submodule of* $g \otimes \mathbb{K}[t]$ *over* $\mathbb{K}[t]$
2. *Find minimal basis* $g_1, \dots, g_n$ *of this module (leading coeffs are lin.ind.)*
3. *Degrees of* g_i = *Kronecker indices, coeffs of* g_i = *canonical basis.*

1. $Z = < d_{X-tA}f_1, \dots, d_{X-tA}f_n >$ *where* f_i – *basis invariants.*
2. *Leading coeffs of* $d_{X-tA}f_i = d_A f_i$. *They are lin.ind.*
3. $m_i = \deg f_i - 1$, *basis* = $\{coeffs\ over\ t\ of\ df_i(X - tA), \forall i\}$

Jordan part

$2x2$ *blocks with* $\lambda = \infty$

Extend the basis of $Ker\ B_A \cap Kron$ *to the basis of the Lagrangian subspace of* $Ker\ B_A$ *with respect to the form B.*



Example, $g = gl_n$

$gl_n = Mat_{nxn}(\mathbb{C})$

$(X,Y) = tr(XY), [X,Y] = XY - YX, ind\ g = n$

Let $\phi = trX^2, \psi = tr\ AX$

$d\phi = tr\ (dX \cdot 2X) \leftrightarrow 2X$

$d\psi = tr\ (dX \cdot A) \leftrightarrow A$

$\{\phi, \psi\} = (X, [2X, A]) = (A, [X, 2X]) = 0$

$\{\phi, \psi\}_A = (A, [2X, A]) = (2X, [A, A]) = 0$

Basis invariants : $f_k = tr\ X^k, df_k = kX^{k-1}$

$Z = < E, X - tA, \dots, (X - tA)^{n-1} >_{\mathbb{K}[t]}$

A – *regular* $\Leftrightarrow \forall \lambda\ \exists!$ *Jordan block*

$E, A, A^2, \dots, A^{n-1}$

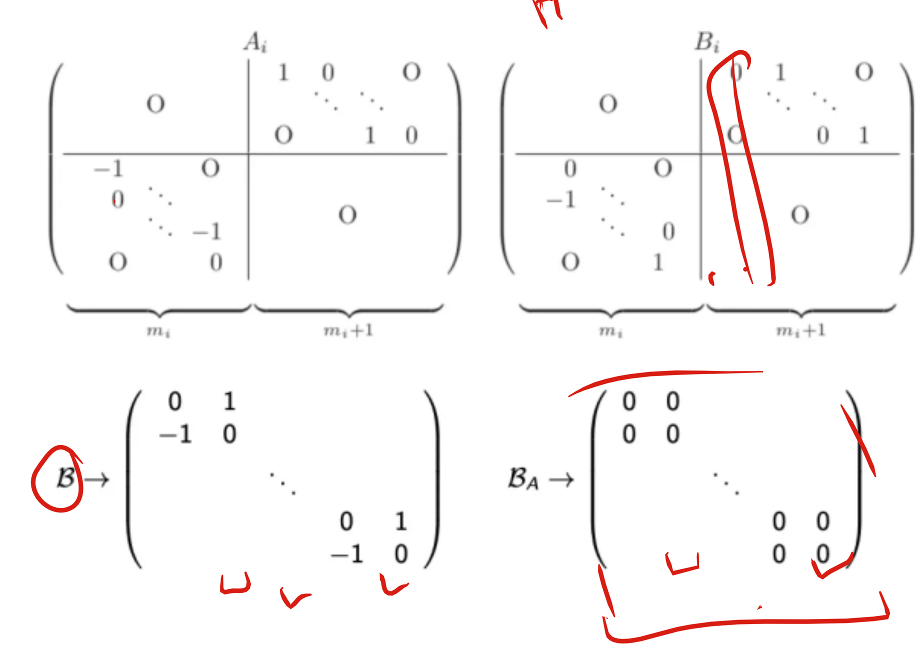
Alg

b_A *has no invariants,* (b_A, b_C) *has JK invariants*

if A – fixed, C – from open dense subset then

JK invariants of (b_A, b_C) = *JK invariants of* (B_A, B)

$rk B_A = 2(m_0 + \dots + m_{\dots})$



Limits

Thm. Constructed complete system of functions in bi – involution freely generate the limit subalgebra $\lim_{s \to 0} F_{A+Bs}$

$\chi_{X-tA}(z) = z^n + c_0 z^{n-1} + \dots + c_{n-1}$

$\chi_{X-tA}(X - tA) = 0$

$C_k(z) = z^k + c_0 z^{k-1} + \dots + c_{k-1} \qquad k = 0 \dots n-1$

Thm. if A – nilpotent, then $\{C_k(X - tA), k = 0 \dots n-1\}$ – *minimal basis of Z.*

Observation : $C_{k-1}(z)$ = *quotient when we divide* $C_k(z)$ *by z.*

$Q_n(z) = \chi_{X-tA}(z) \ \big|\ \frac{z + \mu_{n-1}t}{Q_{n-1}(z)}$

$r_{n-1}(t) \ \big|\ \frac{z + \mu_{n-2}t}{Q_{n-2}(z)}$

r_{n-2}

$\big|\ \frac{z + \mu_0 t}{Q_0(z)}$

r_0

Thm. For any A $\{Q_k(X - tA)\ k = 0 \dots n-1\}$ – *minimal basis of Z*

Nilpotent A

$\chi_{X-tA}(z) = z^n + c_0 z^{n-1} + c_1 z^{n-1} + \dots + c_{n-2} \cdot z + c_{n-1}$

$c_0(X - tA) \quad c_1(X - tA) \quad c_{n-2}(X - tA) \quad c_{n-1}(X - tA)$

$\boxed{c_{0,0}} \quad c_{1,0} \quad c_{n-2,0} \quad c_{n-1,0}$

$\boxed{c_{1,1}} \quad c_{n-2,1} \quad c_{n-1,1}$

$\boxed{c_{n-2,m_{n-2}}} \quad \boxed{c_{n-1,m_{n-1}}}$

Kronecker part

$\boxed{c_{k,m_k}}$ *are basis invariants of* $Ker\ B_A$

$c_{n-2,m_{n-2}}(X - sB) \quad c_{n-2,m_{n-2}}(X - sB)$

$c_{n-2,m_{n-2}+1} \quad c_{n-1,m_{n-1}+1}$

$c_{n-2,n-2} \quad c_{n-1,n-1}$

Jordan part