Set-theoretic Yang-Baxter equation, twists & quandle Hopf algebras

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Review

- YBE introduced: *Yang*, study of N particle in δ potential & *Baxter*, study of XYZ model.
- Fundamental equ. in QISM formulation [*Faddeev, Tahktajan, Sklyanin, Kulish, Reshetikhin....*] & quantum algebras [*Drinfeld, Jimbo*]
- [*Drinfeld*] introduced the **Set-theoretic YBE**.
- Connections to: *braid theory, Hopf algebras, knot theory, low dimensional topology, Hopf-Galois extensions, ternary structures, such as heaps & trusses ...*

Review

- [*Hietiranta*] first to find examples of set-theoretic solutions. [*Etingof, Shedler & Soloviev*] set-theoretic solutions & quantum groups for param. free R-matrices.
- Connections to: geometric crystals [*Berenstein & Kazhdan, Etingof*] and cellular automatons [*Hatayama, Kuniba & Takagi*]. Etingof rational solutions from geometric crystal theory.
- Classical discrete integrable systems (YB maps), quad-graph, discrete maps, solitons interactions: [*Veselov, Bobenko, Suris, Papageorgiou, Tongas,...*] **Parametric!**
- Set-theoretic involutive solutions of YBE from **braces**: [*Rump, Guarnieri, Vendramin, Gateva-Ivanova, Cedó, Jespers, Okniński, Smoktunowicz,...*]

Talk outline

- I will discuss the algebraic approach for solving the set-theoretic YBE, basic blueprint in [*AD, Rybolowicz, Stefanelli*] (parametric case [*AD*])
- Preliminaries and motivations. Introduce the set-theoretic YBE and the notions of shelves, racks and quandles.
- Admissible Drinflel'd twist: all set-theoretic solutions obtained from shelves (racks) or th flip map and an admissible twist! Prototypical algebraic solutions presented.
- Formulate the underlying quasi-triangular Hopf algebraic structures. Well known examples of quantum algebras: Yangians and q-deformed algebras.
  A new paradigm of Quantum Algebras (especially the parametric case!)

Preliminaries: Set-theoretic braid equation

- Let a set $X = \{x_1, \ldots, x_{\mathcal{N}}\}$ and $\check{r} : X \times X \to X \times X$. Denote $\check{r} : X \times XX \times X$

Solution of the braid equation

$$\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$$

1. $(X, \check{r})$ non-degenerate: σ_x and τ_y are bijective functions
2. $(X, \check{r})$ involutive: $\check{r}(\sigma_x(y), \tau_y(x)) = (x, y)$, $\check{r}^2 = \mathrm{id}$

- *Suppose* $(X, \check{r})$ *is an involutive, non-degenerate set-theoretic solution of the Braid equation*:

$$(\check{r} \times \mathrm{id}_X)(\mathrm{id}_X \times \check{r})(\check{r} \times \mathrm{id}_X) = (\mathrm{id}_X \times \check{r})(\check{r} \times \mathrm{id}_X)(\mathrm{id}_X \times \check{r}).$$

Set-theoretic YBE

- **Remark.** if $\check{r}$ satisfies the set-theoretic braid equation then $R := \check{r}\pi$ (π is the flip map: $\pi(a,b) = (b,a)$ for all $a, b \in X$) satisfies the YBE:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$R(b,a) = (\sigma_a(b), \tau_b(a))$ and $R_{12}(c,b,a) = (\sigma_b(c), \tau_c(b), a)$ ...

- Then, the general solution of the set-theoretic YBE is a map $R : X \times X \to X \times X$, such that

Solution of the YBE

$$R(b,a) = (\sigma_a(b), \tau_b(a))$$

Matrices

- **Linearization:** $x_j \to e_{x_j}$, then $\mathbb{B} = \{e_{x_j}\}$, $x_j \in X$ is a basis of $V = \mathbb{C}X$ space of dimension equal to the cardinality of X. Recall, $e_{x,y} = e_x e_y^T$, $\mathcal{N} \times \mathcal{N}$ matrices. *Set-theoretic $\check{r}$ as $\mathcal{N}^2 \times \mathcal{N}^2$ matrix:*

Matrix form

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$$

Then $\check{r}$ satisfies:

$$(\check{r} \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \check{r})(\check{r} \otimes \mathrm{id}_X) = (\mathrm{id}_X \otimes \check{r})(\check{r} \times \mathrm{id}_X)(\mathrm{id}_X \otimes \check{r}).$$

- **Baxterization for involutive solutions**: $\check{r} : V \otimes V \to V \otimes V$: $\check{r}^2 = I_{V \otimes V}$. Reps of the symmetric group. *Baxterization*:

$$\check{R}(\lambda) = \lambda \check{r} + 1_{V \otimes V}$$

In the special case $\check{r} = \mathcal{P}$ ($\mathcal{P}$: permutation op) we recover the **Yangian**.
If $\lambda = 0$ then $\check{r} = 1_{V \otimes V} \to$ commuting Hamiltonians!

Local Hamiltonians

- Results by [*AD & Smoktunowicz*].

Local Hamiltonian

$$H = \sum_{n=1}^{N} \sum_{x,y\in X} e^{(n)}_{x,\sigma_x(y)}\ e^{(n+1)}_{y,\tau_y(x)}$$

Unlike Yangian, periodic Ham is *not* $\mathfrak{gl}_N$ **symmetric...***Surprise!*
(twisted Yangian coproduts, quasi bialgebra!).
Lyubashenko solution, $\sigma(y) = y+1,\ \tau(x) = x-1,\ mod\mathcal{N},\ x, y \in \{1, 2, \dots, \mathcal{N}\}$,

$$\boxed{H = \sum_{n=1}^{N} \sum_{x,y=1}^{\mathcal{N}} e^{(n)}_{x,y+1}\ e^{(n+1)}_{y,x-1}}$$

.

- Spectrum and eigenstates of commuting Hams challenging (symmetries of transfer matrix). Deriving Drinfeld twist key steps [*AD*] (non-local maps [*Soloviev*])!
- q-deformed version of the involutive set-theoretic solutions has been constructed via an admissible Drinfeld twist [*AD & Smoktunowicz*].

Shelves, racks & quandles

- Focus on special non-involutive set-theoretic solutions $\check{r}(x, y) = (y, y \triangleright x)$, where $\triangleright : X \times X \to X$, some binary operation.
- Shelves, racks & quandles [*Joyce, Matveev, Dehornoy,....*] satisfy axioms analogous to the Reidemeister moves used to manipulate knot diagrams. Link invariants, coloring of links a knot is tri-colored or not; Alexander's theorem: all links closed braids.

Definition

Let X be a non-empty set and $\triangleright$ a binary operation on X. Then, the pair $(X, \triangleright)$ is said to be a *left shelf* if $\triangleright$ is left self-distributive, namely, the identity

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$$

is satisfied, for all $a, b, c \in X$. Moreover, a left shelf $(X, \triangleright)$ is called

1. a *left rack* if $a\triangleright$ is bijective, for every $a \in X$.
2. a *quandle* if $(X, \triangleright)$ is a left rack and $a \triangleright a = a$, for all $a \in X$.

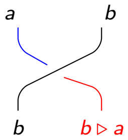
Shelves, racks & quandles

1. **Conjugate quandle.** Let $(X, \cdot)$ be a group and $\triangleright : X \times X \to X$, such that $a \triangleright b = a^{-1} \cdot b \cdot a$. Then $(X, \triangleright)$ is a quandle.
2. **Core quandle:** Let $(X, \cdot)$ be a group and $\triangleright : X \times X \to X$, such that $a \triangleright b = a \cdot b^{-1} \cdot a$. Then $(X, \triangleright)$ is a quandle.
3. **Affine (or Alexander) quandle.** Let X be a non empty set equipped with two group operations, $+$ and $\circ$. Define $\triangleright : X \times X \to X$, such that for $z \in X$ and $\forall\ a, b \in X$, $a \triangleright b = -a \circ z + b \circ z + a$. Similar to a $\mathbb{Z}(t, t^{-1})$ ring module. (*For non-abelian* $(X, +)$ [*AD, Stefanelli, Rybolowicz*]).

Proposition

Let X be a non empty set, then the map $\check{r} : X \times X \to X \times X$, such that $\check{r}(a, b) = (b, b \triangleright a)$ is a solution of the braid equation if and only if $(X, \triangleright)$ is a shelve. The solution is invertible if and only if $(X, \triangleright)$ is a rack.

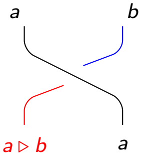
- Solutions from quandles **non-involutive**! All non-involutive set-theoretic solutions come from quandles by admissible Drinfeld twist [*AD, Rybolowicz, Stefanelli*].
- Extra motivation: q-deformed racks, quandles....from q braids.



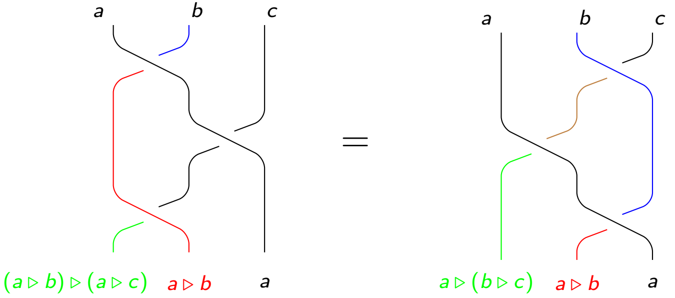
$$\check{r} = \sum_{a,b\in X} e_{a,b} \otimes e_{b,b\triangleright a}$$

- $\check{r}^{-1}(a, b) = (a \triangleright^{-1} b, a)$, $\check{r}(a, b) = (a \triangleright b, a)$ also solution of braid equ.

Self-distributivity - shelve solutions



$\check{r} = \sum_{a,b \in X} e_{a,a \triangleright b} \otimes e_{b,a}$



$$(\check{r} \times id)(id \times \check{r})(\check{r} \times id) \quad = \quad (id \times \check{r})(\check{r} \times id)(id \times \check{r})$$

Examples of quandles

- Let $i, j \in X := \{1, 2, \ldots, n\}$ and define $i \triangleright j = 2i - j \ mod \ n$: $(X, \triangleright)$ is a quandle called the **dihedral quandle** (a core quandle).
- Special case [*Dehornoy*]. $n = 3$, $X = \{x_1, x_2, x_3\}$, $\triangleright : X \times X \to X$, such that:

$\triangleright$	x_1	x_2	x_3
x_1	x_1	x_3	x_2
x_2	x_3	x_2	x_1
x_3	x_2	x_1	x_3

- The 3D vector space. The canonical basis:

$$\hat{e}_{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{e}_{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{e}_{x_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Recall $\check{r} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,y \triangleright x}$, where $e_{x,y}$ the elementary 3×3 matrix $e_{x,y} = e_x e_y^T$.
I.e. $\check{r} = \sum_{j=1}^{3} e_{x_j,x_j} \otimes e_{x_j,x_j} + e_{x_1,x_2} \otimes e_{x_2,x_3} + e_{x_2,x_1} \otimes e_{x_1,x_3} + \dots$

The $\check{r}$ matrix:

$$\check{r} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\check{r}^{-1} = \check{r}^T$. Unitary quantities from Twisted Yangian, [*AD*] in progress.
- Combinatorial matrices! [*Kauffman...*]: **qudits, topological quantum computing - braid gates.**

KEY STATEMENTS.

1. All involutive set-theoretic solutions of the braid equation, $\check{r} = \sum_{a,b \in X} e_{a,\sigma_a(b)} \otimes e_{b,\tau_b(a)}$ come from the permutation operator via an *admissible Drilfenl'd twist* (similarity) [*AD*].
2. All generic **non**-involutive set-theoretic solutions $\check{r}$ come from quandle solutions operator via an *admissible Drilfenl'd twist* [*AD, Stefanelli, Rybolowicz*]. *Generalized in the parametric case* [*AD*].

Generic solutions

- We focus on *generic* solutions of the set-theoretic YBE, $\check{r} : X \times X \to X \times X$, such that for all $a, b \in X$,

$$\check{r}(a,b) = (\sigma_a(b), \tau_b(a))$$

- In this case, biracks and biquandles (**two binary operations**): virtual links & braids (ribbons).
- Generic solutions obtained via admissible Drinfeld twist!!

Proposition

Let X, be a non-empty set, and define for all $a, b \in X$, the maps $\sigma_a, \tau_b : X \to X$, $b \mapsto \sigma_a(b)$ and $a \mapsto \tau_b(a)$. Then $\check{r} : X \times X \to X \times X$, such that for all $a, b \in X$, $\check{r}(a,b) = (\sigma_a(b), \tau_b(a))$ is a solution of the set-theoretic braid equation if and only if

$$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$$
$$\tau_c(\tau_b(a)) = \tau_{\tau_c(b)}(\tau_{\sigma_b(c)}(a))$$
$$\sigma_{\tau_{\sigma_b(c)}(a)}(\tau_c(b)) = \tau_{\sigma_{\tau_b(a)}(c)}(\sigma_a(b)).$$

Skew braces

- Introduce fundamental useful algebraic structures.

Definition (skew braces)

[*Rump, Guarnieri & Vendramin*] A *left skew brace* is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that for all $a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

If $+$ is an abelian group operation B is called a *left brace*. Moreover, if B is a left skew brace and for all $a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a *two sided skew brace.*

- The additive identity of B will be denoted by 0 and the multiplicative identity by 1. In every skew brace $0 = 1$. Braces $\to$ radical rings [*Rump, Smoktunowicz,...*]! From now on when we say skew brace we mean left skew brace.

Examples of braces

Example

1. Finite braces. Let $U(\mathbb{Z}/2^n\mathbb{Z}) =: U_n$ denote a set of odd integers mod 2^n, $n \in \mathbb{N}$. Define also $+_1 : U_n \times U_n \to U_n$, such that $a +_1 b := a - 1 + b$, for all $a, b \in U_n$. Moreover, $+$ is the usual addition and $\circ$ is the usual multiplication of integers. Then the triplet $(U_n, +_1, \circ)$ is a brace. For instance: 1. $n = 1$, $U_1 = \{1\}$, 2. $n = 2$, $U_2 = \{1,\ 3\}$, 3. $n = 3$, $U_2 = \{1,\ 3,\ 5,\ 7\}$ ...

Example

2. Infinite braces. Consider a set $O := \{\frac{2n+1}{2k+1} | n, k \in \mathbb{Z}\}$ together with two binary operations $+_1 : O \times O \to O$ such that $(a, b) \mapsto a - 1 + b$ and $\circ : O \times O \to O$ such that $(a, b) \mapsto a \circ b$, where $+$, $\circ$ are addition and multiplication of rational numbers, respectively. Then the triplet $(O, +_1, \circ)$ is a brace

Solutions from skew braces

Proposition (Rump - Guarnieri & Vendramin)

Let $(X, +, \circ)$ be a skew brace and define $\sigma_a : X \to X$, such that $\sigma_a(b) = -a + a \circ b$ and $a \circ b = \sigma_a(b) \circ \tau_b(a)$. Then $\check{r} : X \times X \to X \times X$, such that $\check{r}(a, b) = (\sigma_a(b), \tau_b(a))$ is a solution of the set-theoretic braid equation.

- **Remark.** If $(X, +, \circ)$ is a brace (Rump), i.e. $(X, +)$ is abelian, $\check{r}$ is involutive, $\check{r}^2 = \text{id}$.

  All involutive set-theoretic solutions $\check{r}$ are obtained from the flip map via an admissible twist (the corresponding solutions of the YBE are obtained from the identity map.)

Admissible Drinfel'd twists

Definition

Let $(X, \check{r})$ and $(X, \check{s})$ be solutions of the set-theoretic braid equation. We say that a map $\varphi : X \times X \to X \times X$ is a *Drinfel'd twist* (*D-twist*) if

$$\varphi\, \check{r} = \check{s}\, \varphi,$$

If φ is a bijection we say that $(X, \check{r})$ and $(X, \check{s})$ are *D-equivalent (via φ)*, and we denote it by $\check{r} \cong_D \check{s}$.

Proposition

Let $(X, \check{r})$ be a left non-degenerate solution, such that for $a, b \in X$, $\check{r}(a, b) = (\sigma_a(b), \tau_b(a))$ and let $(X, \check{s})$ be a solution, such that for $a, b \in X$, $\check{s}(a, b) = (b, b \triangleright a)$ and $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(\sigma_a(b) \triangleright a)$. Then $\check{r}$ is D-equivalent to $\check{s}$.

- **Proof.** Let $\varphi : X \times X \to X \times X$ be the map defined by $\varphi(a, b) := (a, \sigma_a(b))$, for all $a, b \in X$, (φ is bijective). Then

$$\varphi^{-1}\ \check{s}\ \varphi(a, b) = \ldots = (\sigma_a(b), \tau_b(a)) = \check{r}(a, b),$$

where $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(\sigma_a(b) \triangleright a)$. That is $\check{r} \cong_D \check{s}$.

- **Remark.** In the special case of involutive $\check{r}$-matrices we observe that $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$, which leads to $b \triangleright a = a$, and hence $\check{s}(a, b) = (b, a)$ for all $a, b \in X$, i.e. $\check{s} = \pi$, i.e. the flip map.

Admissible twists & general solutions

Definition

Let $(X, \triangleright)$ be a shelf. We say that the twist $\varphi : X \times X \to X \times X$, such that $\varphi(a,b) := (a, \sigma_a(b))$ for all $a, b \in X$, is admissible, if for all $a, b, c \in X$:
$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$ & $\sigma_c(b) \triangleright \sigma_c(a) = \sigma_c(b \triangleright a)$.

Theorem

Let $(X, \triangleright)$ be a shelf and $\varphi : X \times X \to X \times X$, such that $\varphi(a,b) := (a, \sigma_a(b))$ for all $a, b \in X$. Then, the map $\check{r} : X \times X \to X \times X$ defined by

$$\check{r}\,(a,b) = \Big(\sigma_a\,(b)\,,\ \sigma^{-1}_{\sigma_a(b)}(\sigma_a(b) \triangleright a)\Big)$$

for all $a, b \in X$, is a solution of the braid equ. if and only if φ is an admissible twist.

Proof. The proof is quite involved based on the (1), (2) of the Definition of the adm. twist and the three fundamental relations from the braid equation.

Corollary 1.

Any left non-degenerate solution $\check{r} : X \times X \to X \times X$, $\check{r}(a, b) = (\sigma_a(b), \tau_b(a))$, for all $a, b \in X$, is obtained from a shelve solution, where $a \triangleright b = \sigma_a(\tau_{\sigma_b^{-1}(a)}(b))$, via an admissible twist.

Corollary 2.

A left non-degenerate solution $(X, \check{r})$ is bijective if and only if $(X, \triangleright)$ is a rack.

- **Conclusion.** The problem of finding generic solutions of the set-theoretic braid equation is reduced to the classification of shelve/rack solutions and the identification of admissible twists.
- For $\check{r}$ being involutive it suffices to find for all $a \in X$, a bijective map $\sigma_a : X \to X$ such that, $\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$.

Solutions from quandles via twists

We assume the existence of the bijective map $\sigma_a : X \to X$ and $(X, +, \circ)$ is a skew brace.

1. **From the conjugate quandle.** This case corresponds to latter Proposition. $\sigma_a(b) = -a + a \circ b$ provides a solution to the YBE. Also, $a \circ b = \sigma_a(b) \circ \tau_b(a)$ (*Guarnieri-Vendramin* solution).
2. **From the affine quandle.** $\sigma_a(b) = -f(a) + a \circ b$, where $f(a) := a \circ z - z$, $z \in X$ is a fixed element. Also, $a \circ b = \sigma_a(b) \circ \tau_b(a)$ (deformed solutions *AD & Rybolowicz*).
3. **From the core quandle.** $\sigma_a(b) = a + a \circ b$. σ_a provides a solution of the YBE if and only if $(X, +)$ is *abelian* group. Also, $a \circ b = \sigma_a(b) \circ \tau_b(a)$ (*AD*).

Part II: Hopf algebras

- Recall linearization: tensor products
  1. $R = \sum_{a,d \in X} e_{b,\sigma_a(b)} \otimes e_{a,\tau_b(a)}$, generic set-theoretic solutions:
  2. $R = \sum_{a,b \in X} e_{b,a} \otimes e_{a,b \triangleright a}$, shelve solutions,
- We establish the algebraic framework in the tensor product formulation. This naturally provides solutions to set-theoretic YBE, thus the linearized version is essential in what follows.
- Next, explore algebraic structures that provide universal $\mathcal{R}$-matrices associated to rack and general set-theoretic solutions of the YBE.

Rack algebras

Definition

Let X be a non-empty set. We define the binary operation, $\triangleright : X \times X \to X$, $(a,b) \mapsto a \triangleright b$. Let also $(X, \triangleright)$ be a finite magma, or such that $a\triangleright$ is surjective, for every $a \in X$. We say that the unital, associative algebra $\mathcal{Q}$, over a field k generated by, $1_{\mathcal{Q}}$, q_a, $(q_a^{-1}$, $h_a \in \mathcal{Q}$ $(h_a = h_b \Leftrightarrow a = b)$ and relations for all $a, b \in X$:

$$q_a q_a^{-1} = q_a^{-1} q_a = 1_{\mathcal{Q}}, \quad q_a q_b = q_b q_{b\triangleright a},$$
$$h_a h_b = \delta_{a,b} h_a, \quad q_b h_{b\triangleright a} = h_a q_b$$

is a rack algebra.

The choice of the name *rack algebra* is justified by the following result.

Proposition

Let $\mathcal{Q}$ be the rack algebra, then for all $a, b, c \in X$ $c \triangleright (b \triangleright a) = (c \triangleright b) \triangleright (c \triangleright a)$, i.e. $(X, \triangleright)$ is a rack.

Proof. Compute $h_a q_b q_c$ using **associativity** and invertibility of q_a for all $a \in X$,:

$$h_{c\triangleright(b\triangleright a)} = h_{(c\triangleright b)\triangleright(c\triangleright a)} \;\Rightarrow\; c \triangleright (b \triangleright a) = (c \triangleright b) \triangleright (c \triangleright a).$$

$a\triangleright$ is bijective, thus $(X, \triangleright)$ is a rack.

The universal R-matrix

Proposition

Let $\mathcal{Q}$ be the rack algebra and $\mathcal{R} \in \mathcal{Q} \otimes \mathcal{Q}$ be an invertible element, such that $\mathcal{R} = \sum_a h_a \otimes q_a$. Then $\mathcal{R}$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

$\mathcal{R}_{12} = \sum_{a\in X} h_a \otimes q_a \otimes 1_{\mathcal{Q}}$, $\mathcal{R}_{13} = \sum_{a\in X} h_a \otimes 1_{\mathcal{Q}} \otimes q_a$, and $\mathcal{R}_{23} = \sum_{a\in X} 1_{\mathcal{Q}} \otimes h_a \otimes q_a$. The inverse $\mathcal{R}$-matrix is $\mathcal{R}^{-1} = \sum_{a\in X} h_a \otimes q_a^{-1}$.

Proof. From YBE and rack algebra relations. Also, $\mathcal{R}^{-1} = \sum_{a\in X} h_a \otimes q_a^{-1}$.

- **Fundamental representation:** Recall, $e_{i,j}$, $n \times n$ matrices with elements $(e_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. Let $\mathcal{Q}$ be the rack algebra and $\rho : \mathcal{Q} \to \mathrm{End}(V)$, defined by $q_a \mapsto \sum_{x\in X} e_{x,a\triangleright x}$, $\quad h_a \mapsto e_{a,a}$. Then $\mathcal{R} \mapsto R = \sum_{a,b\in X} e_{b,b} \otimes e_{a,b\triangleright a}$: the linearized rack solution.

Quandle Hopf algebras

Definition

A rack algebra $\mathcal{Q}$ is called a quandle algebra if there exits a left quasigroup $(X, \bullet)$, such that $a \bullet b = b \bullet (b \triangleright a)$, for all $a, b \in X$.

Theorem

Let $\mathcal{A}$ be the quandle algebra with $(X, \bullet, e)$ being a group. Let also $\mathcal{R} = \sum_{a \in X} h_a \otimes q_a$ be a solution of the Yang-Baxter equation and $q_a q_b = q_{a \bullet b}$ for all $a, b \in X$. Then the structure $(\mathcal{A}, \Delta, \epsilon, S, \mathcal{R})$ is a quasi-triangular Hopf algebra:

- Co-product. $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta(q_a^{\pm 1}) = q_a^{\pm 1} \otimes q_a^{\pm 1}$ and $\Delta(h_a) = \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet c = a}$.
- Co-unit. $\epsilon : \mathcal{A} \to k$, $\epsilon(q_a^{\pm 1}) = 1$, $\epsilon(h_a) = \delta_{a,e}$.
- Antipode. $S : \mathcal{A} \to \mathcal{A}$, $S(q_a^{\pm 1}) = q_a^{\mp 1}$, $S(h_a) = h_{a^*}$, where a^* is the inverse in $(X, \bullet)$ for all $a \in X$.

- Relevant: Pointed Hopf Algebras from racks [*Andruskiewitsch & Grana*].

Proof (quandle quasi-triangular Hopf algebra). Show all the axioms of a quasi-triangular Hopf algebra. First,

$$\mathcal{R}_{13}\mathcal{R}_{12} = \sum_{a\in X} h_a \otimes q_a \otimes q_a =: \sum_{a\in X} h_a \otimes \Delta(q_a) =: (\mathrm{id}\otimes\Delta)\mathcal{R},$$
$$\mathcal{R}_{13}\mathcal{R}_{23} = \sum_{a,b\in X} h_a \otimes h_b \otimes q_c\Big|_{a\bullet b=c} =: \sum_{c\in X} \Delta(h_c)\otimes q_c =: (\Delta\otimes\mathrm{id})\mathcal{R},$$

read of $\Delta(h_a)$, $\Delta(q_a)$ as:

$$\Delta(q_a^{\pm 1}) = q_a^{\pm 1}\otimes q_a^{\pm 1}, \quad \Delta(h_a) = \sum_{b,c\in X} h_b\otimes h_c\Big|_{b\bullet c=a}.$$

$\Delta : \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is an algebra homomorphism, checked via the distributivity condition $a \triangleright (b\bullet c) = (a\triangleright b)\bullet(a\triangleright c)$, which follows from, $a\triangleright b = a^*\bullet b\bullet a$.

Moreover,

$$\Delta^{(op)}(q_a^{\pm 1})\mathcal{R} = \mathcal{R}\Delta(q_a^{\pm 1}) \quad \Delta^{(op)}(h_a)\mathcal{R} = \mathcal{R}\Delta(h_a),$$

$\Delta^{(op)} = \pi\circ\Delta$, π is the flip map.

Proof. Check co-associativity and uniquely derive the counit $\epsilon : \mathcal{A} \to k$ (homomorphism) and antipode $S : \mathcal{A} \to \mathcal{A}$ (anti-homomorphism).

1. *Co-associativity*: $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.

$$(\mathrm{id} \otimes \Delta)\Delta(q_a) = (\Delta \otimes \mathrm{id})\Delta(q_a) = q_a \otimes q_a \otimes q_a,$$
$$(\mathrm{id} \otimes \Delta)\Delta(h_a) = (\Delta \otimes \mathrm{id})\Delta(h_a) = \sum_{b,c,d \in X} h_b \otimes h_c \otimes h_d \Big|_{b \bullet c \bullet d = a}.$$

2. *Counit*: $(\epsilon \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \epsilon)\Delta(x) = x,$ for all $x \in \{q_a,\ q_a^{-1}, h_a\}$. The generators q_a are group-like elements, so $\epsilon(q_a) = 1$, and

$$\sum_{a,b \in X} \epsilon(h_a) h_b = \sum_{a,b} h_a \epsilon(h_b)\Big|_{a \bullet b = c} = h_c \Rightarrow \epsilon(h_a) = \delta_{a,e}.$$

3. *Antipode*: $m\big((S \otimes \mathrm{id})\Delta(x)\big) = m\big((\mathrm{id} \otimes S)\Delta(x)\big) = \epsilon(x) 1_{\mathcal{A}}$ for all $x \in \{q_a,\ q_a^{-1}, h_a\}$.
For q_a, we immediately have $S(q_a) = q_a^{-1}$ and (recall $h_a h_b = \delta_{a,b} h_a$ and $\sum_{a \in X} h_a = 1_{\mathcal{A}}$)

$$\sum_{a,b \in X} S(h_a) h_b \Big|_{a \bullet b = c} = \sum_{a,b \in X} h_a S(h_b)\Big|_{a \bullet b = c} = \delta_{c,e} 1_{\mathcal{A}} \ \Rightarrow\ S(h_a) = h_{a^*},$$

where a^* is the inverse in $(X, \bullet)$ for all $a \in X$. $(\mathcal{A}, \Delta, \epsilon, S, \mathcal{R})$ is indeed a quasi-triangular Hopf algebra.

Co-associativity

- The n-coproducts $\Delta^{(n)} : \mathcal{A} \to \mathcal{A}^{\otimes n}$, such that for all $a_1, a_2, \dots, a_n \in X$,

$$\Delta^{(n)}(q^{\pm 1}) = q_a^{\pm 1} \otimes q_a^{\pm 1} \otimes \ldots \otimes q_a^{\pm 1},$$
$$\Delta^{(n)}(h_a) := \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes h_{a_2} \otimes \ldots \otimes h_{a_n}\Big|_{a_1 \bullet a_2 \bullet \ldots \bullet a_n = a}.$$

- **Remark (no co-associativity).** Let $(X, \triangleright)$ be a quandle and $(X, \bullet)$ be a magma with a left neutral element, such that $a \bullet b = b \bullet (b \triangleright a)$. $\mathcal{A}$ is the quandle algebra, $\mathcal{R} = \sum_{a \in X} h_a \otimes q_a$ is the associated universal $\mathcal{R}$-matrix and $q_a, F_a : X \to \mathcal{A}$, $q_a q_b = F_{a \bullet b}$ and $a \triangleright (b \bullet c) = (a \triangleright b) \bullet (a \triangleright c)$. Then $(\mathcal{A}, \Delta, \epsilon, \mathcal{R})$ is a quasi-triangular quasi-bialgebra as no co-associativity holds [*AD, Rybolowicz, Stefanelli*]!

The decorated rack algebra

Definition

Let $\mathcal{Q}$ be the rack algebra. Let also $\sigma_a,\ \tau_b : X \to X$, and σ_a be a bijection for all $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\hat{\mathcal{Q}}$ over k, generated by intederminates $q_a, q_a^{-1}, h_a, \in \mathcal{Q}$ and $w_a, w_a^{-1} \in \hat{\mathcal{Q}}$, $a \in X$, $1_{\hat{\mathcal{Q}}} = 1_{\mathcal{Q}}$ is the unit element and relations, for $a, b \in X$,

$$q_a q_a^{-1} = q_a^{-1} q_a = 1_{\hat{\mathcal{Q}}}, \quad q_a q_b = q_b q_{b \triangleright a}, \quad h_a h_b = \delta_{a,b} h_a,$$
$$q_b h_{b \triangleright a} = h_a q_b \quad w_a (w_a)^{-1} = 1_{\hat{\mathcal{Q}}}, \quad w_a w_b = w_{\sigma_a(b)} w_{\tau_b(a)}$$
$$w_a h_b = h_{\sigma_a(b)} w_a, \quad w_a q_b = q_{\sigma_a(b)} w_a$$

is a *decorated rack algebra*.

Proposition.

Let $\hat{\mathcal{Q}}$ be the decorated rack algebra, then for all $a, b, c \in X$:

$$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) \quad \& \quad \sigma_c(b) \triangleright \sigma_c(a) = \sigma_c(b \triangleright a).$$

Proof. Follow from the algebra associativity. **These are the conditions of an admissible twist!**

Proposition.

Let $\hat{\mathcal{Q}}$ be the decorated rack algebra and $\mathcal{R} = \sum_a h_a \otimes q_a \in \mathcal{Q} \otimes \mathcal{Q}$ be the universal $\mathcal{R}$-matrix. We also define $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,

$$\Delta((y_a)^{\pm 1}) := (y_a)^{\pm 1} \otimes (y_a)^{\pm 1}, \quad \Delta(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet c = a}.$$

$y_a \in \{q_a, \ w_a\}$.

Then the following statements hold:

1. Δ is a $\hat{\mathcal{Q}}$ algebra homomorphism.
2. $\mathcal{R}\Delta(y_a) = \Delta^{(op)}(y_a)\mathcal{R}$, for $y_a \in \{q_a, \ w_a\}$, $a \in X$. Recall, $\Delta^{(op)} := \pi \circ \Delta$, where π is the flip map.

Universal R-matrix by twisting

- **Proposition.** Let $\mathcal{R} = \sum_{a\in X} h_a \otimes q_a \in \mathcal{Q} \otimes \mathcal{Q}$ be the rack universal $\mathcal{R}-$matrix, $\hat{\mathcal{Q}}$ be the decorated rack algebra and $\mathcal{F} \in \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$, $\mathcal{F} = \sum_{b\in X} h_b \otimes (w_b)^{-1}$, then $\mathcal{F}$ is an admissible twist.

  This guarantees that if $\mathcal{R}$ is a solution of the YBE then $\mathcal{R}^F$ also is!

- The twisted $\mathcal{R}-$matrix:

$$\mathcal{R}^F = \mathcal{F}^{(op)} \mathcal{R} \mathcal{F}^{-1}.$$

- The twisted coproducts: $\Delta^F(y) = \mathcal{F}\Delta(y)\mathcal{F}^{-1}$, $y \in \hat{\mathcal{Q}}$. Moreover it follows that $\mathcal{R}\Delta^F(y) = \Delta^{F(op)}(y)\mathcal{R}^F$, $y \in \hat{\mathcal{Q}}$.

- **Fundamental representation & the set-theoretic solution:**
  Let $\hat{\mathcal{Q}}$ be the decorated p-rack algebra, $\rho : \hat{\mathcal{Q}} \to \mathrm{End}(V)$, such that

$$q_a \mapsto \sum_{x\in X} e_{x,a\triangleright x}, \quad h_a \mapsto e_{a,a}, \quad w_a \mapsto \sum_{b\in X} e_{\sigma_a(b),b},$$

  then $\mathcal{R}^F \mapsto R^F = \sum_{a,b\in X} e_{b,\sigma_a(b)} \otimes e_{a,\tau_b(a)}$, where $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(\sigma_a(b) \triangleright a)$.

  R^F is the linearized version of the set-theoretic solution.

- The associated quantum algebra (non-parametric case) is a *quasi-triangular quasi bialgebra* [*AD, Vlaar, Ghionis*].