Maximal Poisson commutative subalgebras and 2-splittings of semisimple Lie algebras

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Complete integrability

Fundamental Problem in Theory of Integrable Systems:

Construct a complete involutive set of first integrals for a Hamiltonian dynamical system.

Algebraic version:

Construct Poisson commutative subalgebras of maximal possible transcendence degree in Poisson algebras.

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Poisson algebras

Ground field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition

A *Poisson algebra* is a unital commutative associative algebra A equipped with a bilinear map $\{\cdot,\cdot\} : A \times A \to A$ (*Poisson bracket*) such that:

- $(A, \{\cdot,\cdot\})$ is a Lie algebra;
- *Leibniz rule*: $\{f, gh\} = \{f, g\} \cdot h + g \cdot \{f, h\}$, $\forall f, g, h \in A$.

Typical example: (M, π) is a Poisson manifold,
π is a Poisson bivector field, $A = \mathcal{F}(M, \mathbb{K})$, $\{f, g\} = \pi(df, dg)$.

Particular case: $\mathfrak{g}$ is a Lie algebra, $M = \mathfrak{g}^*$, $A = \mathbb{K}[\mathfrak{g}^*] = \mathrm{S}(\mathfrak{g})$,
Lie–Poisson bracket: $\{\xi, \eta\} = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$,
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Typical example: (M, π) is a Poisson manifold,
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Commutative subalgebras

Let G be a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$.

Symplectic leaves in $\mathfrak{g}^*$ = coadjoint orbits $G \cdot \gamma, \quad \gamma \in \mathfrak{g}^*$.
Kirillov–Kostant–Souriau symplectic form on $G \cdot \gamma$:

$$\omega_\gamma(\xi \cdot \gamma, \eta \cdot \gamma) = \langle [\xi, \eta], \gamma \rangle, \qquad \forall \xi, \eta \in \mathfrak{g}.$$

Index $\quad \operatorname{ind} \mathfrak{g} = \operatorname{codim}_{\mathfrak{g}^*}(G \cdot \gamma)$,
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Problem

Construct a subalgebra $C \subset A = \mathrm{S}(\mathfrak{g})$ such that:

- $\{C, C\} = 0$;
- $\operatorname{tr.deg} C = \max = \frac{1}{2} \dim(G \cdot \gamma) + \operatorname{codim}(G \cdot \gamma) \qquad$ *($\gamma \in \mathfrak{g}^*$ gen. pt.)*
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Lenard–Magri scheme

Let $\{\cdot,\cdot\}'$ be another Poisson bracket on A compatible with $\{\cdot,\cdot\}$
$\rightsquigarrow$ pencil of Poisson brackets $\{\cdot,\cdot\}_t = \{\cdot,\cdot\}' + t\{\cdot,\cdot\}$, $t \in \mathbb{K}$

Poisson center $Z_t := \{f \in A \mid \{f,g\}_t = 0,\ \forall g \in A\}$

$$\operatorname{tr.deg} Z_t \le \dim \operatorname{Ker}(\pi_t)_\gamma \quad (\gamma \in \mathfrak{g}^* \text{ gen. pt.})$$

Properties:

1. $\{Z_{t_1}, Z_{t_2}\}_t = 0$ for $t = t_1, t_2$ $(t_1 \neq t_2)$ $\implies$ for any $t \in \mathbb{P}^1$;
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   Proof: take $\mathbb{P}^1_{\mathrm{reg}} \ni t_2 \to t_1$ and apply (1).

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Often happens: $\operatorname{tr.deg} C = \max$

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Example 1 (frozen argument)

$\{\xi,\eta\}' = \langle[\xi,\eta],\gamma_0\rangle \in \mathbb{K}, \quad \forall \xi,\eta\in\mathfrak{g}, \quad \gamma_0\in\mathfrak{g}^*$ fixed.

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(*Mishchenko–Fomenko argument shift method*)

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Proof of Theorem 1

Semisimple Lie algebras are rigid $\implies \mathfrak{g}_t \simeq \mathfrak{g}$, $\forall t \in \mathbb{P}^1 \setminus$ finite set $=: \mathbb{P}^1_0$.

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Jordan–Kronecker normal form:

- Kronecker blocks of sizes $(2m_i + 1) \times (2m_i + 1) \quad (i = 1, \dots, r)$;
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Semisimple Lie algebras are rigid $\implies \mathfrak{g}_t \simeq \mathfrak{g},\ \forall t \in \mathbb{P}^1 \setminus \text{finite set} =: \mathbb{P}^1_0$.

$\mathfrak{g}_t^{\text{sing}} := \{\gamma \in \mathfrak{g}^* \mid \dim(G_t \cdot \gamma) < \max = \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}_t\}$,

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Jordan–Kronecker normal form:

- Kronecker blocks of sizes $(2m_i + 1) \times (2m_i + 1) \quad (i = 1, \ldots, r)$;
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Jordan–Kronecker normal form

Kronecker blocks:

$$
\left(\begin{array}{cccc|ccccc}
 & & & & 1 & t & & & \\
 & & & & & 1 & t & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & 1 & t \\
\hline
-1 & & & & & & & & \\
-t & -1 & & & & & & & \\
 & -t & \ddots & & & & & & \\
 & & \ddots & -1 & & & & & \\
 & & & -t & & & & &
\end{array}\right)
$$

$$\underbrace{\qquad\qquad}_{m_i} \quad \underbrace{\qquad\qquad}_{m_i+1}$$

Jordan blocks:

$$
\left(\begin{array}{cccc|cccc}
 & & & & \lambda_j+t & 1 & & \\
 & & & & & \lambda_j+t & \ddots & \\
 & & & & & & \ddots & 1 \\
 & & & & & & & \lambda_j+t \\
\hline
-\lambda_j-t & & & & & & & \\
-1 & -\lambda_j-t & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -1 & -\lambda_j-t & & & &
\end{array}\right)
$$

$$\underbrace{\qquad\qquad}_{n_j} \quad \underbrace{\qquad\qquad}_{n_j}$$

2-decompostions

Definition (Panyushev–Yakimova)

2-decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$, where $\mathfrak{h}, \mathfrak{f} \subset \mathfrak{g}$ are Lie subalgebras.

2-decomposition $\rightsquigarrow$ pencil of Lie algebra structures on $\mathfrak{g}$:

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Inönü–Wigner contraction of $\mathfrak{g}$ at $\mathfrak{h}$

$\mathfrak{g}_\infty \simeq (\mathfrak{g}/\mathfrak{f})^{\mathrm{ab}} \oplus \mathfrak{f}$, Inönü–Wigner contraction of $\mathfrak{g}$ at $\mathfrak{f}$

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Index of Inönü–Wigner contraction

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$\rightsquigarrow$ pencil of Lie algebras $\mathfrak{g}_t = (\mathfrak{g}, [\cdot,\cdot]_t)$ $(t \in \mathbb{P}^1)$

$\rightsquigarrow$ pencil of linear Poisson brackets $\{\cdot,\cdot\}_t$ on $A = \mathrm{S}(\mathfrak{g})$

$\rightsquigarrow$ Poisson commutative subalgebra $C \subset A$ (by Lenard–Magri scheme).

Note: $\mathfrak{g}_t \simeq \mathfrak{g}$, $\forall t \neq 0, \infty \implies \mathbb{P}^1 \setminus \{0,\infty\} \subset \mathbb{P}^1_{\mathrm{reg}}$

Corollary (of Theorem 1)

$\operatorname{tr.deg} C = \max \iff 0,\infty \in \mathbb{P}^1_{\mathrm{reg}} \iff \operatorname{ind}\mathfrak{g}_0 = \operatorname{ind}\mathfrak{g}_\infty = \operatorname{ind}\mathfrak{g}$ $(= \operatorname{rk}\mathfrak{g})$

Problem

Compute index of Inönü–Wigner contraction.

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Index of Inönü–Wigner contraction

Assume: $\mathfrak{g}$ semisimple, with 2-decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$

$\rightsquigarrow$ pencil of Lie algebras $\mathfrak{g}_t = (\mathfrak{g}, [\cdot,\cdot]_t)$ $(t \in \mathbb{P}^1)$

$\rightsquigarrow$ pencil of linear Poisson brackets $\{\cdot,\cdot\}_t$ on $A = \mathrm{S}(\mathfrak{g})$

$\rightsquigarrow$ Poisson commutative subalgebra $C \subset A$ (by Lenard–Magri scheme).

Note: $\mathfrak{g}_t \simeq \mathfrak{g}$, $\forall t \neq 0, \infty \implies \mathbb{P}^1 \setminus \{0,\infty\} \subset \mathbb{P}^1_{\mathrm{reg}}$

Corollary (of Theorem 1)

$\operatorname{tr.deg} C = \max \iff 0,\infty \in \mathbb{P}^1_{\mathrm{reg}} \iff \operatorname{ind} \mathfrak{g}_0 = \operatorname{ind} \mathfrak{g}_\infty = \operatorname{ind} \mathfrak{g} \ (= \operatorname{rk} \mathfrak{g})$

Problem

Compute index of Inönü–Wigner contraction.

Main results

Note: complexification preserves index $\Longrightarrow$ may assume $\mathbb{K} = \mathbb{C}$.

Assume: $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathfrak{f} = \mathrm{Lie}(F)$, $H, F \subset G$ algebraic subgroups
$\rightsquigarrow X = G/H$, homogeneous algebraic variety

Definition

Complexity $c(X) := \mathrm{codim}_X(B \cdot x)$, $x \in X$ gen. pt., $B \subset G$ Borel subgroup

X (resp. H, $\mathfrak{h}$) is a *spherical* variety (subgroup, subalgebra) if $c(X) = 0$.

Theorem 2 (Panyushev (2007), Panyushev–Yakimova (2021), T. (2025))

$$\mathrm{ind}\, \mathfrak{g}_0 = \mathrm{rk}\, \mathfrak{g} + 2c(G/H)$$

Corollary

$\mathrm{tr.deg}\, C = \max \iff \mathfrak{h}, \mathfrak{f} \subset \mathfrak{g}$ *are spherical subalgebras*

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X affine

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X affine | *X quasi-affine*

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Raïs Formula

Suppose: $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{f}$ *Lie algebra,* $\mathfrak{h} \subset \mathfrak{q}$ *Lie subalgebra,* $\mathfrak{f} \lhd \mathfrak{q}$ *Abelian ideal.*
Then: $\operatorname{ind} \mathfrak{q} = \operatorname{codim}_{\mathfrak{f}^*}(H \cdot \gamma) + \operatorname{ind} \mathfrak{h}_\gamma, \quad \gamma \in \mathfrak{f}^*,$ *general point.*

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Knop (1990): $\operatorname{codim}_{T^*X}(G \cdot \gamma) = 2c(X) + r(X)$

Here: *rank* $r(X) := \dim(G \cdot \mathfrak{h}^\perp)/G, \quad \mathfrak{g}^* \supset \mathfrak{h}^\perp \simeq (\mathfrak{g}/\mathfrak{h})^*.$

Knop (1990), Zhgoon (2013), T. (2025): description of $\mathfrak{h}_\gamma$.

Namely: $\mathfrak{g}^* \simeq \mathfrak{g} \supset \mathfrak{l}$ (Levi subalgebra) $\supset \mathfrak{s} \supset [\mathfrak{l}, \mathfrak{l}]$ such that
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Examples

Example 1: $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$.
Here: $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{b}^- =$ opposite Borel subalgebra, $\mathfrak{n} =$ nilradical of $\mathfrak{b}^-$.

E.g.: $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{b} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$, $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\}$

$A^G = \mathbb{K}[f_2, \ldots, f_n]$, $f_k(\gamma) = \mathrm{tr}(\gamma^k)$

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Some references

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