Functional realization of the Gelfand-Tselin base

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Irreducible representations of $\mathfrak{gl}_n$

V — a finite dimentional irreducible representation of $\mathfrak{gl}_n$.

$$\mathfrak{gl}_n = \langle E_{i,j} \rangle_{i,j=1,\dots,n} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$
$$\mathfrak{g}_- = \langle E_{i,j} \rangle_{i>j}, \quad \mathfrak{h} = \langle E_{i,i} \rangle, \quad \mathfrak{g}_+ = \langle E_{i,j} \rangle_{i<j}.$$

A weight vector is an eigenvector for all $E_{i,i}$: $E_{i,i}v = \lambda_i v$. It's weight is $[\lambda_1, \dots, \lambda_n]$.

Theorem

There exists a unique highest weight vector $v \in V$ i.e. a weight vector annihilated by $\mathfrak{g}_+$: $\forall i < j : \; E_{i,j}v = 0$.
The weight of a highest weight vector (the highest weight) $[m_1, \dots, m_n]$ has the property

$$m_i - m_{i+1} \in \mathbb{Z}_{\geq 0}.$$

Classification of Irreducible Representations

Next, we restrict ourselves to representations with $m_n = 0$. Then the highest weight has the form

$$m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0, \quad m_i \in \mathbb{Z}. \tag{1}$$

Theorem

For each set of numbers satisfying (1), there exists a unique finite-dimensional irreducible representation with that highest weight.

Gelfand–Tsetlin base: the case of $\mathfrak{gl}_3$

Let $V_{[m_1,m_2,0]}$ be a representation of $\mathfrak{gl}_3$.

Restrict $\mathfrak{gl}_3 =< E_{i,j} >_{i,j=1,2,3}$ to $\mathfrak{gl}_2 =< E_{i,j} >_{i,j=1,2}$:

$$\mathfrak{gl}_3 \downarrow \mathfrak{gl}_2 \quad \Rightarrow \quad V_{[m_1,m_2,0]} = \bigoplus V_{[k_1,k_2]}$$

Restrict $\mathfrak{gl}_2$ to $\mathfrak{gl}_1 =< E_{1,1} >$:

$$\mathfrak{gl}_2 \downarrow \mathfrak{gl}_1 \quad \Rightarrow \quad V_{[k_1,k_2]} = \bigoplus V_{h_1}.$$

This corresponds to the Gelfand–Tsetlin diagramm:

$$V_{h_1} =< v >, \quad v = \begin{pmatrix} m_1 & & m_2 & & 0 \\ & k_1 & & k_2 & \\ & & h_1 & & \end{pmatrix}$$

Zhelobenko Model

We consider functions on the group GL_n.

$$(Gf)(g) := f(gG), \quad g, G \in GL_n, \quad f(g) \in \mathrm{Fun}(GL_n)$$

Examples of functions:

1. $a_i^j \in \mathrm{Fun}(GL_n)$ — matrix element (i is the column index, j is the row index)
2. $a_{i_1,\dots,i_k} := \det(a_i^j)_{i=i_1,\dots,i_k}^{j=1,\dots,k}$

Then

$$E_{p,q} \cdot a_{i_1,\dots,i_k} = a_{i_1,\dots,i_k|_{q\mapsto p}}$$

In particular, the function

$$a_1^{m_1-m_2} a_{1,2}^{m_2-m_3} \dots a_{1,2,\dots,n-1}^{m_{n-1}}$$

is a highest weight vector of weight $[m_1, \dots, m_{n-1}, 0]$.

Zhelobenko Model

Theorem (Zhelobenko)

Polynomials in the determinants $a_{i_1,\dots,i_k}$ form a model for the representations of $\mathfrak{gl}_n$.
The representation with the highest weight $[m_1,\dots,m_{n-1},0]$ consists of those polynomials for which the sum of the degrees of the i-th order determinants equals $m_i - m_{i+1}$.

Important remark: there are many relations among the determinants a_X, $X \subset \{1,\dots,n\}$. For example:

$$a_1 a_{2,3} - a_2 a_{1,3} + a_3 a_{1,2} = 0$$

What function on GL_n corresponds to the Gelfand-Tselin base vectors?

Theorem (Biedenharn, Baird 1963)

In the case $n = 3$ to the Gelfand-Tselin diagramm

$$\begin{pmatrix} m_1 & & m_2 & & 0 \\ & k_1 & & k_2 & \\ & & h_1 & & \end{pmatrix}$$

there corresponds the function

$$a_3^{m_1-k_1} a_{1,2}^{k_2} a_1^{s-m_2} a_2^{k_1-s} a_{1,3}^{m_2-k_2} \cdot$$
$$\cdot F_{2,1}(s-k_1, k_2-m_2, s-m_2+1; \frac{a_1 a_{2,3}}{a_2 a_{1,3}})$$

Γ-Series

$z = \{z_1, \ldots, z_N\}$ — a set of variables.

$\mathbb{Z}^N$ — the lattice of exponents of monomials in the variables z.

γ — a fixed vector, $L \subset \mathbb{Z}^N$ — a sublattice.

$$\mathcal{F}_\gamma(z) = \sum_{x \in \gamma + L} \frac{z^x}{\Gamma(x+1)}$$

Let $F_{2,1}(a_1, a_2, b_1; z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(a_1)_n (a_2)_n}{(b_1)_n} z^n$, where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, be the Gauss hypergeometric series.

Then if $\gamma = (-a_1, -a_2, b_1 - 1, 0)$, and $B = \mathbb{Z}\langle(-1, -1, 1, 1)\rangle = \mathbb{Z}\langle v \rangle$, we have

$$\mathcal{F}_\gamma(z_1, z_2, z_3, z_4) = c\, \underbrace{z_1^{-a_1} z_2^{-a_2} z_3^{b_1 - 1}}_{z^\gamma} F_{2,1}\left(a_1, a_2, b_1; \underbrace{\frac{z_3 z_4}{z_1 z_2}}_{z^v} \right)$$

$$c = \frac{1}{\Gamma(1 - a_1)\Gamma(1 - a_2)\Gamma(b_1)}$$

The GKZ System

1. Let $a = (a_1, \dots, a_N)$ be a vector orthogonal to the lattice B. Then

$$a_1 z_1 \frac{\partial}{\partial z_1}\mathcal{F}_\gamma + \dots + a_N z_N \frac{\partial}{\partial z_N}\mathcal{F}_\gamma = (a_1\gamma_1 + \dots + a_N\gamma_N)\mathcal{F}_\gamma,$$

and it suffices to consider only the basis vectors of the lattice orthogonal to B.
2. Let $b \in B$ and write $b = b_+ - b_-$. Isolate the nonzero elements in these vectors: $b_+ = (\dots, b_{i_1}, \dots, b_{i_k}, \dots)$, $b_- = (\dots, b_{j_1}, \dots, b_{j_l}, \dots)$. Then

$$\mathcal{O}_v\mathcal{F}_\gamma = \left(\left(\frac{\partial}{\partial z_{i_1}}\right)^{b_{i_1}} \dots \left(\frac{\partial}{\partial z_{i_k}}\right)^{b_{i_k}} - \left(\frac{\partial}{\partial z_{j_1}}\right)^{b_{j_1}} \dots \left(\frac{\partial}{\partial z_{j_l}}\right)^{b_{j_l}}\right)\mathcal{F}_\gamma = 0$$

For example, in the case $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, $B = \mathbb{Z}\langle(-1, -1, 1, 1)\rangle$, we have

$$\left(z_1 \frac{\partial}{\partial z_1} + z_3 \frac{\partial}{\partial z_3}\right) \mathcal{F}_\gamma = (\gamma_1 + \gamma_3)\mathcal{F}_\gamma$$

$$\left(z_1 \frac{\partial}{\partial z_1} + z_4 \frac{\partial}{\partial z_4}\right) \mathcal{F}_\gamma = (\gamma_1 + \gamma_4)\mathcal{F}_\gamma$$

$$\left(z_2 \frac{\partial}{\partial z_2} + z_4 \frac{\partial}{\partial z_4}\right) \mathcal{F}_\gamma = (\gamma_2 + \gamma_4)\mathcal{F}_\gamma$$

$$\left(\frac{\partial^2}{\partial z_1 \partial z_2} - \frac{\partial^2}{\partial z_3 \partial z_4}\right) \mathcal{F}_\gamma = 0$$

The Gelfand–Tsetlin Lattice: Motivation

Suppose a function

$$\mathcal{F} = \sum_X c_X a_X^{r_X}$$

corresponds to a Gelfand–Tsetlin pattern. What can its exponents be?

Case $n = 3$: a naive approach

$$a_1^{r_1} a_2^{r_2} a_3^{r_3} a_{1,2}^{r_{1,2}} a_{1,3}^{r_{1,3}} a_{2,3}^{r_{2,3}} \quad v.s. \quad \begin{pmatrix} m_1 & & m_2 & & 0 \\ & k_1 & & k_2 & \\ & & h_1 & & \end{pmatrix}$$

$\mathfrak{gl}_3$-maximization: $a_1, a_2, a_3 \mapsto a_1, \quad a_{1,2}, a_{1,3}, a_{2,3} \mapsto a_{1,2} \Rightarrow$

$$r_1 + r_2 + r_3 + r_{1,2} + r_{1,3} + r_{2,3} = m_1,$$
$$r_{1,2} + r_{1,3} + r_{2,3} = m_2$$

$\mathfrak{gl}_2$-maximization: $a_1, a_2 \mapsto a_1, \quad a_3 \mapsto 0, \quad a_{1,3}, a_{2,3} \mapsto a_{1,3}, a_{1,2} \mapsto a_{1,2} \Rightarrow$

$$r_1 + r_2 + r_{1,2} + r_{1,3} + r_{2,3} = k_1,$$
$$r_{1,2} = k_2$$

$\mathfrak{gl}_1$-maximization: $a_1 \mapsto a_1, \quad a_{1,2} \mapsto a_{1,2}$, other $\mapsto 0 \Rightarrow$:

$$r_1 + r_{1,2} + r_{1,3} = h_1$$

$$a_1^{r_1} a_2^{r_2} a_3^{r_3} a_{1,2}^{r_{1,2}} a_{1,3}^{r_{1,3}} a_{2,3}^{r_{2,3}} \quad v.s. \quad \begin{pmatrix} m_1 & & m_2 & & 0 \\ & k_1 & & k_2 & \\ & & h_1 & & \end{pmatrix}$$

A naive answer:

$$\begin{cases} r_1 + r_2 + r_3 + r_{1,2} + r_{1,3} + r_{2,3} = m_1, \\ r_{1,2} + r_{1,3} + r_{2,3} = m_2, \\ r_1 + r_2 + r_{1,2} + r_{1,3} + r_{2,3} = k_1, \\ r_{1,2} = k_2, \\ r_1 + r_{1,2} + r_{1,3} = h_1 \end{cases}$$

This defines a shifted lattice $\gamma + B$, $\quad B = \mathbb{Z}\langle e_1 - e_2 - e_{1,3} + e_{2,3} \rangle$

The Gelfand–Tsetlin Lattice: Definition

Definition

The Gelfand–Tsetlin lattice is the sublattice of $\mathbb{Z}^{2^n-2}$ defined by the system of equations of the form:

$$v \in B \Leftrightarrow \sum_{X\colon X \text{ has } p \text{ elements } \leq q} r_X = 0$$

Proposition

The lattice B is generated by the vectors

$$v_\alpha := e_{i,X} - e_{j,X} - e_{i,y,X} + e_{j,y,X}, \quad i < j < y$$

Gelfand–Tsetlin Diagramms vs. Shift Vectors

With a Gelfand–Tsetlin diagramm $(m_{i,j})_{j=1,\dots,n,\, i\leq j}$ one can associate a shifted lattice $\Pi = \gamma + B$, where

$$v \in \Pi \Leftrightarrow \sum_{X\colon X \text{ has } p \text{ elements } \leq q} r_X = m_{p,q}$$

Shift vectors $\gamma \mod B \leftrightarrow$ Gelfand–Tsetlin diagramms

Proposition

Π contains a vector with all nonnegative coordinates $\Leftrightarrow$ the betweness conditions hold for $m_{i,j}$.

Theorem (Biedenharn, Baird 1963)

In the case $n = 3$ one has $\mathcal{F}_\gamma(a)$ is a function corresponding to a Gelfand-Tselin diagramm

The direct generalization of this theorem to the case $n > 3$ is not true.

The A-GKZ System

Let A_X, where $X \subset \{1, \ldots, n\}$, be independent variables that are antisymmetric in X. Define the action:

$$E_{p,q} \cdot A_{i_1,\ldots,i_k} = A_{i_1,\ldots,i_k|_{q \mapsto p}}$$

The vector

$$A_1^{m_1 - m_2} A_{1,2}^{m_2 - m_3} \ldots A_{1,2,\ldots,n-1}^{m_{n-1}}$$

is a highest weight vector with weight $[m_1, \ldots, m_{n-1}, 0]$. Which polynomials constitute $V^{[m_1,\ldots,m_{n-1},0]} \subset \mathbb{C}[A]$?

The determinants a_X satisfy the Plucker relations. Let $I \subset \mathbb{C}[A]$ be the ideal generated by these relations.

Make the substitution

$$A_X \mapsto \frac{d}{dA_X},$$

to obtain an ideal $\bar{I} \subset \mathrm{Diff}_{\mathrm{const}}$ which corresponds to a system of PDEs.

Definition

The resulting system of PDEs is called the A-GKZ' system.

The A-GKZ' Model

Theorem

The space of polynomial solutions to the A-GKZ' system is a model for the representations. The representation with highest weight $[m_1, \ldots, m_{n-1}, 0]$ is the space of polynomials such that the sum of the degrees of the A_X with $|X| = i$ equals $m_i - m_{i+1}$.

The A-GKZ System

Introduce the differential operators:

$$\mathcal{O}_\alpha = \frac{\partial^2}{\partial A_{i,X}\partial A_{j,y,X}} - \frac{\partial^2}{\partial A_{j,X}\partial A_{i,y,X}},$$

$$\bar{\mathcal{O}}_\alpha = \mathcal{O}_\alpha + \frac{\partial^2}{\partial A_{y,X}\partial A_{i,j,X}}$$

Definition

The system generated by equations $\bar{\mathcal{O}}_\alpha F = 0$ is called the A-GKZ system.

Theorem

The spaces of polynomial solutions to the A-GKZ and A-GKZ' systems coincide.

Polynomial Solutions of the A-GKZ System

Chose a base among vectors v_α generating the lattice B.
For $\alpha = 1, \dots, k$ and the generating vector

$$v_\alpha := e_{i,X} - e_{j,X} - e_{i,y,X} + e_{j,y,X}$$

define

$$r_\alpha := e_{y,X} - e_{i,j,X} - e_{i,y,X} + e_{j,y,X}$$

For $s \in \mathbb{Z}^k_{\geq 0}$, define

$$J^s_\delta(z) := \sum_{t \in \mathbb{Z}^k} \frac{z^{\delta+tv}(s+1)\dots(s+k-1)}{(\delta+tv)!}$$

Then

$$F_\delta = \sum_{s \in \mathbb{Z}^k_{\geq 0}} \frac{(-1)^s}{s!} J^s_{\delta - sr}(z).$$

Basis of the A-GKZ Model

Theorem

The functions F_δ for vectors δ such that: in the equivalence class modulo B there exists a vector with all coordinates nonnegative form a basis in the space of polynomial solutions of the A-GKZ system.

Define a partial order:

$$\delta \prec \gamma \Leftrightarrow \delta = \gamma + sr \mod B, \quad s \in \mathbb{Z}^k_{\geq 0}$$

If one chooses base in another way $\Rightarrow$ one gets other functions $\tilde{F}_\delta$.

$$\tilde{F}_\delta = \sum_{\gamma \preceq \delta} c_\gamma F_\gamma$$

Relation with the Gelfand–Tsetlin Basis

Theorem

The pairing $\langle F(A), G(A)\rangle := F\left(\frac{d}{dA}\right) G(A)\Big|_{A_X=0}$ is an invariant scalar product in the A-GKZ realization.

Theorem

Orthogonalization of the basis $F_\delta(A)$ with respect to this order (in the decreasing direction) yields the Gelfand–Tsetlin basis.

Connection with the Gelfand–Tsetlin Basis

Theorem

$$F_\delta(A) = \sum_{l \in \mathbb{Z}^k_{\geq 0}} S^l_\delta \cdot G_{\delta - lr}(A), \quad G_\delta(A) = \sum_{l \in \mathbb{Z}^k_{\geq 0}} S^l_\delta \cdot F_{\delta - lr}(A),$$

$$S^0_\delta = \frac{1}{C^0_\delta}, \quad S^l_\delta = -\frac{C^l_\delta}{C^0_\delta C^0_{\delta - lr}}, \quad l \neq 0,$$

$$C^l_\delta = \sum_{u \in \mathbb{Z}^k_{\geq 0}, t \in \mathbb{Z}^k} \frac{(-1)^l (t+1) \dots (t+u+l)(t+1) \dots (t+u)}{(\delta - (l+u)r + tv)!(u+l)!u!}.$$

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