Classifications on soliton solutions of KP type systems

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Background information of KP solitons

The KP equation is given by

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

where $x = t_1, y = t_2, t = t_3$. It is well known that the KP soliton solution $u(x, y, t)$ is given in terms of the τ-function:

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \tau(x, y, t).$$

Exponential solution

The τ-function can be given by the Wronskian form $\tau = \mathrm{Wr}(f_1, \ldots, f_N)$. Each $f_i(t)$ is given by

$$f_i(\mathbf{t}) = \sum_{j=1}^{M} a_{i,j} E_j(\mathbf{t}) \qquad \text{with} \quad E_j(\mathbf{t}) = e^{\xi_j(\mathbf{t})} := \exp(\sum_{n=1}^{\infty} \kappa_j^n t_n),$$

where $\kappa_1 < \cdots < \kappa_M$. Then the τ-function can be rewritten as

$$\tau(\mathbf{t}) = \det(AE^T) \quad \text{with} \quad E(x, y, t) = \begin{pmatrix} E_1 & \cdots & E_M \\ \kappa_1 E_1 & \cdots & \kappa_M E_M \\ \vdots & \ddots & \vdots \\ \kappa_1^{N-1} E_1 & \cdots & \kappa_M^{N-1} E_M \end{pmatrix},$$

where $A := (a_{ij})$ is an $N \times M$ constant matrix and $\mathrm{rank}(A) = N$.

By using the Binet-Cauchy lemma, the τ-function can be expressed by

$$\tau(x,y,t) = \sum_{\mathcal{I}} \Delta_{\mathcal{I}}(A) E_{\mathcal{I}}(x,y,t),$$

with

$$E_{\mathcal{I}}(x,y,t) = \prod_{j>k} (\kappa_{i_j} - \kappa_{i_k}) E_{i_1} E_{i_2} \dots E_{i_N} > 0,$$

where $\Delta_{\mathcal{I}}(A)$ is the $N \times N$ minor for the columns with the index set $\mathcal{I} = \{i_1 < \dots < i_N\}$.

Theorem (Kodama-Williams, Adv. Math. (2013))

Each KP soliton can be determined by a pair of two real data (κ, A), $u(x,y,t)$ is a real regular soliton solution if and only if all the minors are non-negative ($A \in Gr(N,M)_{\geq 0}$, totally nonnegative Grassmannian).

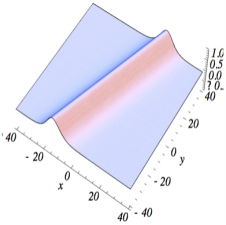
Example

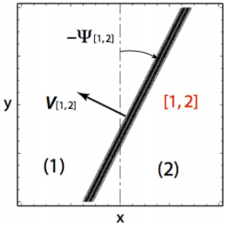
One soliton are from $\kappa_1 < \kappa_2$ and $A = (1, a) \in \mathrm{Gr}(1,2)_{\geq 0}$, we have

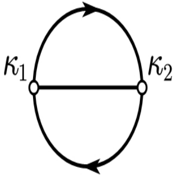
$$\tau = E_1 + aE_2 = e^{\xi_1} + ae^{\xi_2} = 2e^{\frac{\xi_1+\xi_2+\ln a}{2}} \cosh \frac{1}{2}(\xi_1 - \xi_2 - \ln a),$$

where $a > 0$ and $\xi_k = \sum_{n=1}^{\infty} \kappa_k^n t_n = \kappa_k x + \kappa_k^2 y + \kappa_k^3 t + \ldots$, which leads to

$$u(x,y,t) = \frac{(\kappa_1 - \kappa_2)^2}{2} \mathrm{sech}^2 \frac{1}{2}(\xi_1 - \xi_2 - \ln a).$$







The line of the wave crest (or peak) is given by $\xi_1 - \xi_2 - \ln a = 0$, thus the soliton solution is localized along the line $\xi_1 = \xi_2$. $E_1 = e^{\xi_1}$ or $E_2 = e^{\xi_2}$ is the dominant exponential term in the τ-function, and because of this we refer to this solution as a [1,2]-soliton solution.

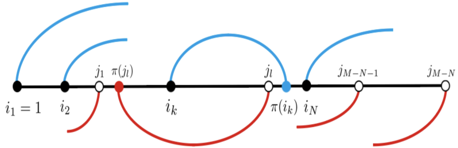
Theorem (Chakravarty-Kodama)

Let $\{i_1, \dots, i_N\}$ be the pivot set and $\{j_1, \dots, j_{M-N}\}$ be the nonpivot set of $A \in Gr(N, M)_{\geq 0}$. Then there exists a unique derangement $\pi(A) \in S_M$ associated with the matrix A, so that the KP soliton has the following asymptotic structure.

- *For $y \gg 0$, there are $[i_n, \pi(i_n)]$-solitons with $\pi(i_n) > i_n$ for $n = 1, \dots, N$;*
- *For $y \ll 0$, there are $[\pi(j_m), j_m]$-solitons with $\pi(j_m) < j_m$ for $m = 1, \dots, M - N$.*

Chord diagram

Assume $A \in \mathrm{Gr}(N, M)_{\geq 0}$, κ_i is a pivot and κ_j is a non-pivot for A matrix, then the chord diagram always has the following structure.



Proposition (Corteel (2007))

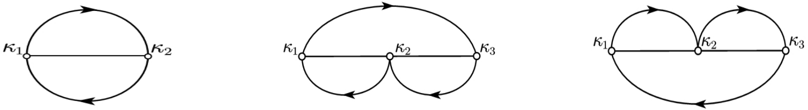
The number of free parameters in $A \in Gr(N, M)_{\geq 0}$ can be found from the chord diagram, and it is given by

$$\{\# \text{ of pivots}\} + \{\# \text{ of crossing}\} + \{\# \text{ of cusps in the lower part}\}.$$

For example, the A matrix of 1-soliton and Y-solitons are given by

$$A_1 = \begin{pmatrix} 1 & a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & a_1 & a_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_2 \end{pmatrix}.$$

The corresponding chord diagrams and permutations are as follows,



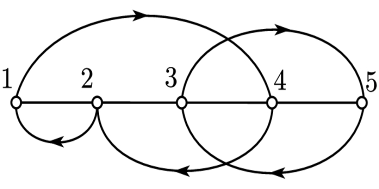
$$\pi(A_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \pi(A_2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \pi(A_3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Example

$$A = \begin{pmatrix} 1 & a_1 & 0 & a_2 & a_3 \\ 0 & 0 & 1 & a_4 & a_5 \end{pmatrix} \in \mathrm{Gr}(2,5)_{\geq 0},$$

$$\pi(A) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} = (1,4,2)(3,5) \quad \text{(cycle notation)}.$$

Its corresponding chord diagram is



$$\dim(A) = 2 + 2 + 1 = 5.$$

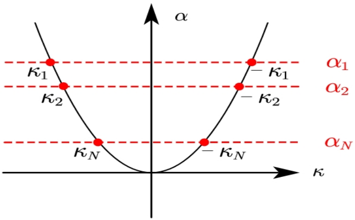
It is well known that the 1-soliton of KP equation is given by

$$u(\mathbf{t}) = \frac{(\kappa_1 - \kappa_2)^2}{2}\operatorname{sech}^2\frac{1}{2}\left((\kappa_i - \kappa_j)x + (\kappa_i^2 - \kappa_j^2)y + (\kappa_i^3 - \kappa_j^3)t\right) - C).$$

The KdV equation as a 2-reduction of the KP hierarchy, which implies $\frac{\partial u}{\partial y} = 0$, this satisfies

$$\kappa_i^2 - \kappa_j^2 = 0 \quad \Rightarrow \quad \kappa_i = \pm\kappa_j.$$

The parabolic is as follows:



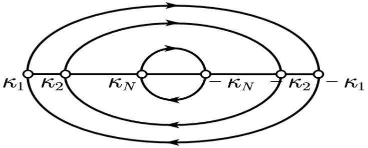
$$(\kappa^2 = \alpha)$$

By choosing $\alpha_1, \ldots, \alpha_N$, we can obtain the N-soliton solution.

$$\kappa = (\kappa_1, \cdots, \kappa_N, -\kappa_N, \cdots, -\kappa_1) \in \mathbb{R}^{2N},$$

$$A = \begin{pmatrix} 1 & & & & & (-1)^{N-1} a_N \\ & \ddots & & & \iddots & \\ & & 1 & a_1 & & \end{pmatrix} \in \mathrm{Gr}(N, 2N)_{\geq 0},$$

where $a_1, \cdots, a_N > 0$. And the chord diagram can be expressed as



The derangement of the N-soliton is given by the product of 2-cycles

$$\pi(A) = \prod_{k=1}^{N} (k, 2N - k + 1).$$

The regular soliton solution of the Boussinesq equation

The (Good) Boussinesq equation as 3-reduction of the KP hierarchy ($\frac{\partial u}{\partial t} = 0$), which can be expressed as

$$(6uu_x + u_{xxx})_x + 3u_{yy} = 0.$$

Here note that this equation does not have a soliton solution with the vanishing boundary condition $u \to 0$ as $|x| \to \infty$. In order to obtain a regular soliton solution, we need a non-vanishing boundary condition.

Assuming $u \to u + u_0$ for $u_0 < 0$, this shifted form of the Boussinesq can be also obtained by the coordinate change in the KP equation,

$$\begin{cases} x \;\to\; x - \dfrac{3}{2} u_0 t_3, \\ t_3 \;\to\; t_3, \end{cases}$$

then the Boussinesq equation can be rewritten in the following standard form

$$u_{yy} - c_0^2 u_{xx} + \frac{1}{3}(3u^2 + u_{xx})_{xx} = 0 \quad \text{with} \quad c_0^2 = -2u_0.$$

We still start from 1-soliton solution of the KP equation

$$u(\mathbf{t}) = \frac{(\kappa_1 - \kappa_2)^2}{2} \mathrm{sech}^2 \frac{1}{2} \left((\kappa_i - \kappa_j) x + (\kappa_i^2 - \kappa_j^2) y + (\kappa_i^3 - \kappa_j^3) t \right) - C).$$

In the new coordinates, the coefficient of the t variable is given by

$$\kappa_i^3 - \kappa_j^3 - \frac{3}{4} c_0^2 (\kappa_i - \kappa_j) = 0 \quad \Rightarrow \quad \kappa_i^3 - \frac{3}{4} c_0^2 \kappa_i = \kappa_j^3 - \frac{3}{4} c_0^2 \kappa_j.$$

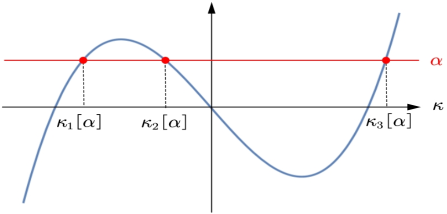
This provides a cubic curve with three real roots

$$\kappa^3 - \frac{3}{4} c_0^2 \kappa - \alpha = (\kappa - \kappa_1)(\kappa - \kappa_2)(\kappa - \kappa_3) = 0.$$

Remark:

$$\begin{cases} \kappa_1 + \kappa_2 + \kappa_3 = 0, \\ \kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3 = -\frac{3}{4}c_0^2, \\ \kappa_1\kappa_2\kappa_3 = \alpha. \end{cases}$$

Then we can describe the solutions of the Boussinesq equation by using cubic curve.



Notice that for each α_i, a matrix $A[\alpha_i] \in \mathrm{Gr}(n_i, m_i)_{\geq 0}$ can be given, where $1 \leq n_i < m_i \leq 3$. For a proper choice of α, we have three real distinct roots as shown below.

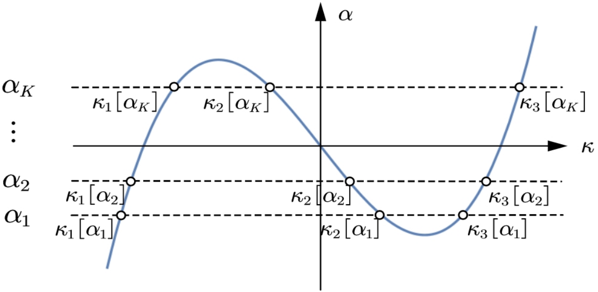
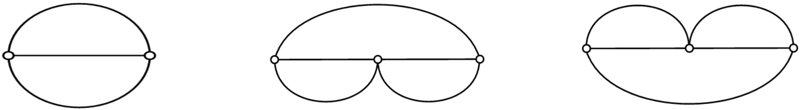


Figure: For each $\alpha = \alpha_j$, the three roots labeled as $(\kappa_1[\alpha_j] < \kappa_2[\alpha_j] < \kappa_3[\alpha_j])$.

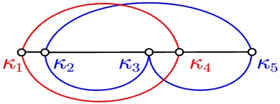
This means that each α_i can provide either a 1-soliton or a Y-soliton.



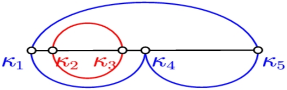
$$A_1 = \begin{pmatrix} 1 & a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & b_1 & b_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & -b_1 \\ 0 & 1 & b_2 \end{pmatrix}.$$

where a_i, b_i are all positive. Then the question is how to combine these 1-solitons and Y-solitons in such a way that the resulting solution remains a regular solution.

For example, can the following combinations yield a regular solution?



$\pi = (1\,,4)\,(2\,,5\,,3)$



$\pi = (1\,,5\,,4)\,(2\,,3)$

$$A_4 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & & & -a_1 & \\ & 1 & b_1 & & b_2 \end{pmatrix} \end{array}, \quad A_5 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & & & -b_1 & -b_2 \\ & 1 & a_1 & & \end{pmatrix} \end{array},$$

It is easy to observe that $\triangle_{45}(A_4) < 0$, which means that A_4 is not totally nonnegative. And all the minors of A_5 are non-negative, i.e., A_5 is totally nonnegative.

We now define the notion of non-crossing of A matrices.

Definition

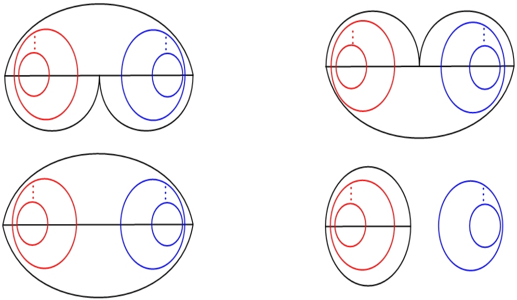
Two matrices are non-crossing if and only if their corresponding chord diagram has no crossing chords.

An immediate consequence of this definition is the following.

Theorem (Huang, Kodama, Li; Commun. Math. Phys. 2025)

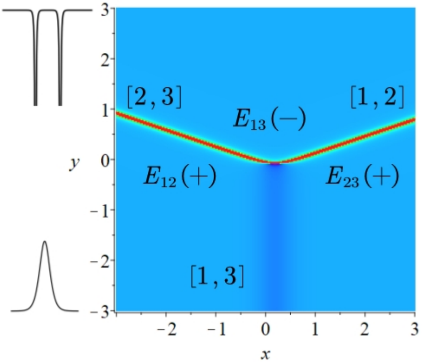
- *If $A[\alpha_i]$ and $A[\alpha_j]$ for $\alpha_i \neq \alpha_j$ are not non-crossing, then the matrix formed by any combination of these two matrices is not totally nonnegative.*
- *Let $A[\alpha_i]$ and $A[\alpha_j]$ for $\alpha_i \neq \alpha_j$ are non-crossing, then we can make the combined matrix becomes totally nonnegative by adjusting the signs in the nonzero entries in $A[\alpha_i]$ and $A[\alpha_j]$.*

Now one can see the general structure. Non-crossing permutation condition provide the following possible case.



Singularity in the Boussinesq solitons

Hirota, Orlov et al. found that the Boussinesq equation has a singular resonant soliton solution with Y-shape, which is a regular soliton for $y \ll 0$ and consists of two singular solitons for $y \gg 0$.



The reason for the emergence of singular solutions is that the A matrix here is given by

$$A = \begin{pmatrix} 1 & 0 & - \\ 0 & 1 & - \end{pmatrix},$$

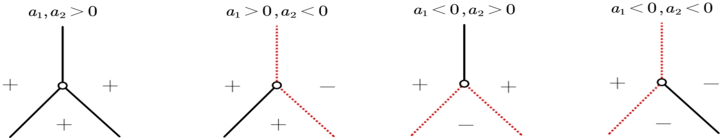
which gives $\Delta_{12} = 1, \Delta_{13} = -1, \Delta_{23} = 1$. For the $[2,3]$-soliton, the τ-function is given by

$$\tau = E_{12} - E_{13} \sim \sinh \frac{1}{2}(\theta_{12} - \theta_{13}).$$

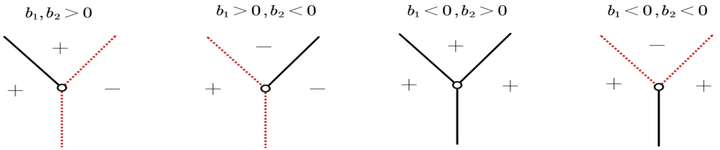
Note that here u can be expressed as

$$u \sim -\operatorname{csch}^2 \frac{1}{2}(\theta_{12} - \theta_{13}).$$

For example, consider $A = \begin{pmatrix} 1 & a_1 & a_2 \end{pmatrix} \in \mathrm{Gr}(1,3)$, we have



And $A = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{pmatrix} \in \mathrm{Gr}(2,3)$, we have



Proposition

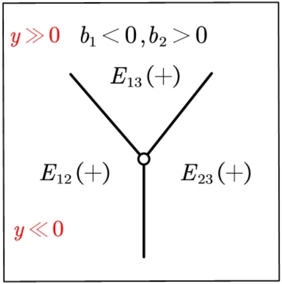
By adjusting the signs of the nonzero entries in A matrix, we can always ensure that the solitons are regular for $y \ll 0$.

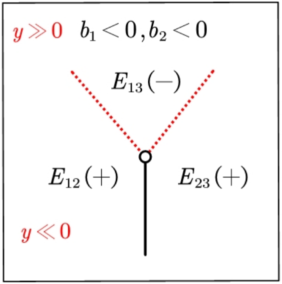
It is noted that as early as 1988, JL. Bona and RL. Sachs proved that the slow solitons $([\kappa_1[\alpha], \kappa_3[\alpha]])$ of the Boussinesq equation are unstable.

Consider the case of $A = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{pmatrix} \in \mathrm{Gr}(2,3)$, the τ-function is given by

$$\tau = (\kappa_2 - \kappa_1)E_{12} - b_1(\kappa_3 - \kappa_2)E_{23} + b_2(\kappa_3 - \kappa_1)E_{13}.$$

Here note that the dominant exponential of the last term is very small when $y \ll 0$. Setting b_1 is negative, if we take b_2 to be positive, we can obtain a regular soliton solution. Conversely, when b_2 is negative, we obtain a singular soliton solution.

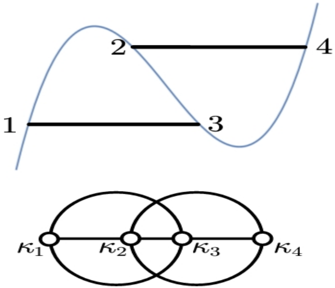


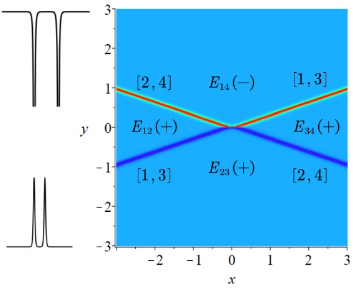


We can treat it as

$$\tau = \tilde{\tau} + \varepsilon \qquad \text{with} \quad \varepsilon \text{ is small.}$$

Question: So, for multiple regular line solitons when $y \to -\infty$, how do the solitons appear when $y \to \infty$?

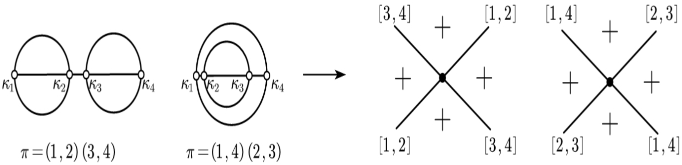




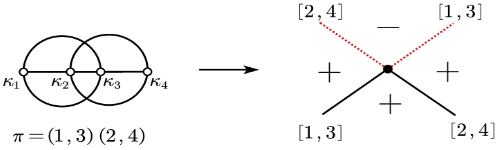
$$A = \begin{pmatrix} 1 & 0 & a_1 & 0 \\ 0 & 1 & 0 & a_2 \end{pmatrix} \qquad \text{with} \quad a_1, a_2 < 0.$$

For the collision of two line solitons, we can categorize them into the following two types.

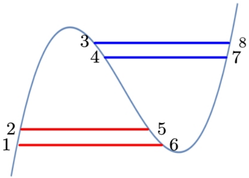
- Non-crossing case (Directly through):



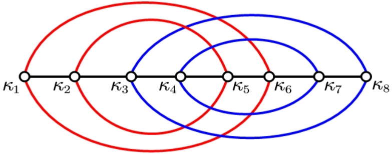
- Crossing case (Transform):



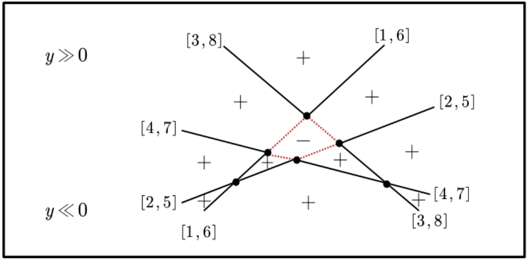
A very interesting example is given by



$\pi = (1,6)\,(2,5)\,(3,8)\,(4,7)$



which gives



The Gel'fand-Dickey ℓ reduction is defined by

$$L^\ell = (L^\ell)_{\geq 0} =: B_\ell = \partial^\ell + v_1 \partial^{\ell-2} + \cdots + v_{\ell-1}.$$

Note that

$$\frac{\partial B_\ell}{\partial t_n} = [B_n, B_\ell] \quad \Rightarrow \quad \frac{\partial v_i}{\partial t_{n\ell}} = 0, \qquad i = 1, \ldots, \ell, \quad n \in \mathbb{Z}^+.$$

Laminate chord

This implies that one can not define the derangement $\pi(A[\alpha_1]\widehat{\oplus}A[\alpha_2])$ in general. However, one can define a laminate chord $\pi(A[\alpha_1]) \cdot \pi(A[\alpha_2])$, superposition of two chords. Let $A[\alpha_1, \alpha_2]$ be the tnn matrix correspond to the laminate chord with $\mathcal{K}_\ell[\alpha_1, \alpha_2]$. This generates a regular soliton with

$$\tau_{A[\alpha_1,\alpha_2]} = \left| A[\alpha_1, \alpha_2] E^T_{[\alpha_1,\alpha_2]} \right|,$$

where

$$A[\alpha_1, \alpha_2] \in \mathrm{Gr}(n_1 + n_2, m_1 + m_2)_{\geq 0}, \quad \pi(A[\alpha_1, \alpha_2]) = \pi(A[\alpha_1]) \cdot \pi(A[\alpha_2]).$$

Non-crossing

We now define the notion of non-crossing of the permutations.

Definition

Let $\pi^{(p)} = (j_1^{(p)}, \ldots, j_{k_p}^{(p)})$ and $\pi^{(q)} = (j_1^{(q)}, \ldots, j_{k_q}^{(q)})$ be two permutations in cycle notation. Then we define

- Two cycle $\pi^{(p)}$ and $\pi^{(q)}$ are non-crossing, if the corresponding chord diagrams have no crossing chords between two permutations.
- Two matrices $A[\alpha_i]$ and $A[\alpha_j]$ for $i \neq j$ are non-crossing, if all the cycles in $\pi(A[\alpha_i])$ are non-crossing with any cycle in $\pi(A[\alpha_j])$.

Proposition: Let $A[\alpha_i] \in \mathrm{Gr}(n_i, m_i)_{\geq 0}$ and $A[\alpha_j] \in \mathrm{Gr}(n_j, m_j)_{\geq 0}$ are non-crossing. Then the κ-direct sum $A[\alpha_i]\widehat{\oplus}A[\alpha_j]$ becomes totally nonnegative by adjusting the signs in the nonzero entries in $A[\alpha_i]$ and $A[\alpha_j]$.

Construction of regular solitons of Gelfand-Dickey systems

Step 1: Choose $\alpha_1 \in \mathbb{R}$ so that

$$\Phi_\ell(\kappa, \alpha_1) = \prod_{j=1}^{\ell} (\kappa - \kappa_j[\alpha_1]) = 0,$$

where $\kappa_i[\alpha_1] \neq \kappa_j[\alpha_1]$ if $i \neq j$. We then define

- $\mathcal{I}[\alpha_1] \subseteq [\ell] = \{1, 2, \cdots, \ell\}$ with $2 \leq |\mathcal{I}[\alpha_1]| = m_1 \leq \ell$;
- $\mathcal{K}[\alpha_1] = \{\kappa_j[\alpha_1] : j \in \mathcal{I}[\alpha_1]\}$;
- $1 \leq n_1 \leq m_1 - 1$.

By taking an element $A[\alpha_1] \in \mathrm{Gr}(n_1, m_1)_{\geq 0}$, whose derangement $\pi(A[\alpha_1])$ is given by the chord diagram.

Step 2: Choose $\alpha_2 \in \mathbb{R}$ so that

$$\Phi_\ell(\kappa, \alpha_2) = \prod_{j=1}^{\ell} (\kappa - \kappa_j[\alpha_2]) = 0.$$

Define

- $\mathcal{I}[\alpha_2] \subseteq [\ell] = \{1, 2, \cdots, \ell\}$ with $2 \leq |\mathcal{I}[\alpha_2]| = m_2 \leq \ell$;
- $\mathcal{K}[\alpha_2] = \{\kappa_j[\alpha_2] : j \in \mathcal{I}[\alpha_2]\}$;
- $1 \leq n_2 \leq m_2 - 1$.

Notice that $A[\alpha_2] \in \mathrm{Gr}(n_2, m_2)_{\geq 0}$ must be under the the condition of non-crossing with $A[\alpha_1]$.

Theorem (Huang, Kodama, Li; Commun. Math. Phys. 2025)

The KP solitons under the ℓ-reduction can be obtained by mutually non-crossing matrices $\{A[\alpha_k] \in Gr(n_k, m_k)_{\geq 0} : k = 1, \cdots, K\}$ and the roots of the polynomials $\Phi_\ell(\kappa, \alpha_k)$ with $k = 1, \cdots, K$. Then we have

- $A[\alpha_1, \ldots, \alpha_K] = A \in Gr(N, M)_{\geq 0}$ *associated with a κ-direct sum of the matrices $A[\alpha_k]$*

$$\widehat{\bigoplus}_{k=1}^{K} A[\alpha_k],$$

  where $N = n_1 + \ldots + n_K, M = m_1 + \ldots + m_K$;
- $\kappa = (\kappa_1, \cdots, \kappa_M) = ord \bigsqcup_{k=1}^{K} \{\kappa_j[\alpha_k] : j \in \mathcal{I}[\alpha_k]\}$;
- $\pi(A) = \prod_{k=1}^{K} \pi(A[\alpha_k])$.

Thank you!