Lagrangian Formulation of the Darboux System

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Hamiltonian and Lagrangian Structures in 2D and in 3D Cases

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Recently we restricted our consideration to the particular case

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we obtained a wide class of interesting new integrable systems. However, some of them belong to a dispersionless limit of the extended Darboux-KP hierarchy. Corresponding Lagrangian density
$$L(u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}, u_{xxx}, u_{xxy}, u_{xxz}, u_{xyy}, u_{xyz}, u_{xzz}, u_{yyy}, u_{yyz}, u_{yzz}, u_{zzz})$$
determines Euler–Lagrange equations of a 6th order.

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determines Euler–Lagrange equations of a 6th order.
In this talk we concentrate to the particular case

$$S=\int L(u_{xy},u_{xz},u_{yz})dxdydz$$

and its integrable dispersive deformations.

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Example:

$$\begin{gathered}\partial_z A(u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}) \\ +\partial_y B(u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}) \\ +\partial_x C(u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}) = 0.\end{gathered}$$

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Emmy Noether Theorem.

4D Integrable Systems and 3D Local Conservation Laws

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$$S=\int L(u_{xx},u_{xy},u_{xz},u_{xt},u_{yy},u_{yz},u_{yt},u_{zz},u_{zt},u_{tt})dxdydzdt.$$

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How many local conservation laws corresponding Euler–Lagrange equation can possess?
We know just two integrable cases:

$$S=\int(u_{xt}+u_{yz}+u_{xx}u_{yy}-u_{xy}^2)^2dxdydzdt;$$

$$S=\int(F\ln F)dxdydzdt,$$

where

$$F=u_{xz}u_{yt}-u_{xy}u_{zt}.$$

Lamé coefficients and Rotation Coefficients

Given a diagonal metric written in terms of the Lamé coefficients H_i,

$$\sum_{i=1}^{n} H_i^2 (dx^i)^2,$$

let us introduce the rotation coefficients β_{ki} via

$$\partial_k H_i = \beta_{ki} H_k,$$

where ∂_k denotes partial derivative with respect to x^k. The requirement that the metric has 'diagonal curvature' (that is, all curvature components $R^i_{kkj} = 0$ for $i \neq j \neq k$), leads to the Darboux system for the rotation coefficients β_{ki},

$$\partial_k \beta_{ij} = \beta_{ik} \beta_{kj},$$

no summation.

Scalar Darboux Potential

It is well-known that under the so-called symmetric reduction, $\beta_{ij} = \beta_{ji}$, Darboux system can be written as a collection of compatible third-order PDEs for a single potential u, one PDE for every triple of distinct indices. This can be achieved by setting $\beta_{ij} = \sqrt{u_{ij}}$ (here and in what follows, lower indices of the potential u indicate partial derivatives), leading to

$$u_{ijk} = 2\sqrt{u_{ij}u_{ik}u_{jk}}.$$

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$$u_{ijk} = 2\sqrt{u_{ij} u_{ik} u_{jk}}.$$

It seems to be less well-known, although explicitly mentioned in Darboux (Chapter III, formula (13)), that the full Darboux system can be represented as a collection of compatible sixth-order PDEs for a single potential u defined via the relations

$$u_{ij} = \beta_{ij}\beta_{ji}.$$

This potential was known to Lamé and Darboux, it was observed later that u is related to the τ-function of KP hierarchy via $u = -\ln\tau$.

Central Scalar Darboux Equation

The sixth-order PDE derived by Darboux is Lagrangian, thus, Darboux system can be written as a collection of compatible sixth-order Lagrangian PDEs for u, one PDE for every triple of distinct indices:

$$\partial_i\partial_j\left(\frac{u_{ijk}+L}{2u_{ij}}\right)+\partial_i\partial_k\left(\frac{u_{ijk}+L}{2u_{ik}}\right)+\partial_j\partial_k\left(\frac{u_{ijk}+L}{2u_{jk}}\right)-\partial_i\partial_j\partial_k\ln(u_{ijk}-L)=0,$$

where $L=\sqrt{u_{ijk}^2-4\,u_{ij}u_{ik}u_{jk}}$. This equation is represented in Euler-Lagrange form corresponding to a third-order Lagrangian, $\int F\,dx^i\,dx^j\,dx^k$, with the Lagrangian density

$$F=L+u_{ijk}\ln(u_{ijk}-L).$$

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We point out that modulo total derivatives, the Lagrangian can be written in several equivalent forms, for instance

$$F = L + \frac{1}{2}u_{ijk}\ln\frac{u_{ijk}-L}{u_{ijk}+L}.$$

Central Scalar Darboux Equation. Derivation

Since

$$u_{ij} = \beta_{ij}\beta_{ji},$$

one can derive the following consequences:

$$u_{ijk} = \beta_{ij}\beta_{jk}\beta_{ki} + \beta_{ik}\beta_{kj}\beta_{ji}.$$

Then we obtain two equations

$$m + n = u_{123}, \quad m \cdot n = u_{12}u_{13}u_{23},$$

where $n = \beta_{12}\beta_{23}\beta_{31}$ and $m = \beta_{13}\beta_{32}\beta_{21}$.

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where $n = \beta_{12}\beta_{23}\beta_{31}$ and $m = \beta_{13}\beta_{32}\beta_{21}$.
Thus,

$$m = \frac{u_{123} - L}{2}, \quad n = \frac{u_{123} + L}{2},$$

where $L = \sqrt{u_{123}^2 - 4\, u_{12}u_{13}u_{23}}$.

Central Scalar Darboux Equation. Derivation

Parametrizing relations

$$u_{ij} = \beta_{ij}\beta_{ji}$$

in the form

$$\begin{array}{ll} \beta_{12} = \sqrt{u_{12}}\, e^{\varphi}, & \beta_{21} = \sqrt{u_{12}}\, e^{-\varphi}, \\ \beta_{13} = \sqrt{u_{13}}\, e^{-\psi}, & \beta_{31} = \sqrt{u_{13}}\, e^{\psi}, \\ \beta_{23} = \sqrt{u_{23}}\, e^{\eta}, & \beta_{32} = \sqrt{u_{23}}\, e^{-\eta}, \end{array}$$

and substituting into the Darboux system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj},$$

we obtain

$$\partial_3 \varphi = \frac{L}{2u_{12}}, \quad \partial_2 \psi = \frac{L}{2u_{13}}, \quad \partial_1 \eta = \frac{L}{2u_{23}},$$

where

$$\varphi + \psi + \eta = \ln \frac{u_{123} - L}{2\sqrt{u_{12}u_{13}u_{23}}}.$$

Central Scalar Darboux Equation. Derivation

Applying to

$$\varphi+\psi+\eta=\ln\frac{u_{123}-L}{2\sqrt{u_{12}u_{13}u_{23}}},$$

the operator $\partial_1\partial_2\partial_3$, one obtains a sixth-order PDE for u,

$$\partial_1\partial_2\frac{L}{2u_{12}}+\partial_1\partial_3\frac{L}{2u_{13}}+\partial_2\partial_3\frac{L}{2u_{23}}-\partial_1\partial_2\partial_3\ln\frac{u_{123}-L}{2\sqrt{u_{12}u_{13}u_{23}}}=0,$$

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$$\partial_1\partial_2 \frac{L}{2u_{12}} + \partial_1\partial_3 \frac{L}{2u_{13}} + \partial_2\partial_3 \frac{L}{2u_{23}} - \partial_1\partial_2\partial_3 \ln \frac{u_{123} - L}{2\sqrt{u_{12}u_{13}u_{23}}} = 0,$$

Remarkably, this equation is in Euler-Lagrange form. To reconstruct the corresponding Lagrangian, note that for a Lagrangian density of the form $F = F(u_{12}, u_{13}, u_{23}, u_{123})$, the corresponding Euler-Lagrange equation is

$$\partial_1\partial_2 \left(\frac{\partial F}{\partial u_{12}}\right) + \partial_1\partial_3 \left(\frac{\partial F}{\partial u_{13}}\right) + \partial_2\partial_3 \left(\frac{\partial F}{\partial u_{23}}\right) - \partial_1\partial_2\partial_3 \left(\frac{\partial F}{\partial u_{123}}\right) = 0.$$

The comparison gives the expressions for all first-order derivatives of F which, on integration, leads to the Lagrangian density

$$F = L + u_{123} \ln \frac{u_{123} - L}{2\sqrt{u_{12}u_{13}u_{23}}}.$$

Differential-difference case (one discrete variable)

Here the starting point is a differential-difference linear system

$$\begin{aligned} \partial_1 H_2 &= \beta_{12} H_1, & \partial_1 H_3 &= \beta_{13} H_1, \\ \partial_2 H_1 &= \beta_{21} H_2, & \partial_2 H_3 &= \beta_{23} H_2, \\ \triangle_3 H_1 &= \beta_{31} H_3, & \triangle_3 H_2 &= \beta_{32} H_3, \end{aligned}$$

where $\triangle_3 = T_3 - 1$ is the discrete x^3-derivative and T_3 denotes unit shift in the discrete variable x^3. The compatibility conditions lead to the differential-difference Darboux system,

$$\begin{aligned} \partial_1 \beta_{23} &= \beta_{21}\beta_{13}, & \partial_1 \beta_{32} &= \beta_{31} T_3 \beta_{12}, \\ \partial_2 \beta_{13} &= \beta_{12}\beta_{23}, & \partial_2 \beta_{31} &= \beta_{32} T_3 \beta_{21}, \\ \triangle_3 \beta_{12} &= \beta_{13}\beta_{32}, & \triangle_3 \beta_{21} &= \beta_{23}\beta_{31}. \end{aligned}$$

Differential-difference case (one discrete variable)

Let us introduce a potential u via the relations

$$u_{12} = \beta_{12}\beta_{21}, \quad \triangle_3 u_1 = \beta_{13}\beta_{31}, \quad \triangle_3 u_2 = \beta_{23}\beta_{32},$$

which are compatible modulo the Darboux system.

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which are compatible modulo the Darboux system.
Introducing the notation $m = \beta_{12}\beta_{23}\beta_{31}$ and $n = \beta_{13}\beta_{32}\beta_{21}$, one has $\triangle_3 u_{12} = m + n + \triangle_3 u_1 \triangle_3 u_2$ and $mn = u_{12}\triangle_3 u_1 \triangle_3 u_2$.

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Solving for m and n one obtains

$$m = \frac{\triangle_3 u_{12} - \triangle_3 u_1 \triangle_3 u_2 - L}{2}, \quad n = \frac{\triangle_3 u_{12} - \triangle_3 u_1 \triangle_3 u_2 + L}{2},$$

where $L = \sqrt{(\triangle_3 u_{12} - \triangle_3 u_1 \triangle_3 u_2)^2 - 4\, u_{12}\triangle_3 u_1 \triangle_3 u_2}$.

Differential-difference case (one discrete variable)

Parametrizing rotation coefficients in the form

$$\begin{aligned}\beta_{12} &= \sqrt{u_{12}}\, e^{\varphi}, \quad \beta_{21} = \sqrt{u_{12}}\, e^{-\varphi},\\ \beta_{13} &= \sqrt{\triangle_3 u_1}\, e^{-\psi}, \quad \beta_{31} = \sqrt{\triangle_3 u_1}\, e^{\psi},\\ \beta_{23} &= \sqrt{\triangle_3 u_2}\, e^{\eta}, \quad \beta_{32} = \sqrt{\triangle_3 u_2}\, e^{-\eta},\end{aligned}$$

we obtain

$$\varphi + \psi + \eta = \ln \frac{m}{\sqrt{u_{12} \triangle_3 u_1 \triangle_3 u_2}}.$$

Differential-difference case (one discrete variable)

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we obtain

$$\varphi+\psi+\eta=\ln\frac{m}{\sqrt{u_{12}\triangle_3 u_1\triangle_3 u_2}}.$$

Under the same parametrization, the Darboux system takes the form

$$\triangle_3\varphi=\frac{1}{2}\ln(1+\frac{\triangle_3 u_{12}}{u_{12}})-\ln(1+\frac{m}{u_{12}}),$$

$$\partial_2\psi=\frac{L}{2\triangle_3 u_1}+\frac{1}{2}\triangle_3 u_2,\quad \partial_1\eta=\frac{L}{2\triangle_3 u_2}-\frac{1}{2}\triangle_3 u_1.$$

Differential-difference case (one discrete variable)

Applying the operator $\partial_1\partial_2\triangle_3$ to

$$\varphi+\psi+\eta=\ln\frac{m}{\sqrt{u_{12}\triangle_3 u_1\triangle_3 u_2}}.$$

one obtains a differential-difference equation in terms of u:

$$\partial_1\partial_2\left(\frac{1}{2}\ln(1+\frac{\triangle_3 u_{12}}{u_{12}})-\ln(1+\frac{m}{u_{12}})\right)$$

$$+\partial_1\triangle_3\left(\frac{L}{2\triangle_3 u_1}+\frac{1}{2}\triangle_3 u_2\right)+\partial_2\triangle_3\left(\frac{L}{2\triangle_3 u_2}-\frac{1}{2}\triangle_3 u_1\right)$$

$$-\partial_1\partial_2\triangle_3\left(\ln\frac{m}{\sqrt{u_{12}\triangle_3 u_1\triangle_3 u_2}}\right)=0.$$

Differential-difference case (one discrete variable)

To reconstruct the corresponding Lagrangian, note that for a Lagrangian density of the form $F = F(u_{12}, \triangle_3 u_1, \triangle_3 u_2, \triangle_3 u_{12})$, the corresponding Euler-Lagrange equation is

$$\partial_1\partial_2\left(\frac{\partial F}{\partial u_{12}}\right) + \partial_1\triangle_3\left(\frac{\partial F}{\partial(\triangle_3 u_1)}\right) + \partial_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_3 u_2)}\right)$$

$$-\partial_1\partial_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_3 u_{12})} - \frac{\partial F}{\partial u_{12}}\right) = 0.$$

Comparison gives the expressions for all first-order derivatives of F which, on integration, leads to the Lagrangian density

$$F = \frac{1}{2}L - u_{12}\ln\left(1 + \frac{m}{u_{12}}\right) - \triangle_3 u_{12}\ln\left(1 + \frac{u_{12}}{m}\right) + \frac{1}{2}\triangle_3 u_{12}.$$

Differential-difference case (two discrete variables)

Here the starting point is a differential-difference linear system

$$\begin{aligned} \partial_1 H_2 &= \beta_{12} H_1, \quad & \partial_1 H_3 &= \beta_{13} H_1, \\ \triangle_2 H_1 &= \beta_{21} H_2, \quad & \triangle_2 H_3 &= \beta_{23} H_2, \\ \triangle_3 H_1 &= \beta_{31} H_3, \quad & \triangle_3 H_2 &= \beta_{32} H_3, \end{aligned}$$

where $\triangle_2$ and $\triangle_3$ denote discrete derivatives in the variables x^2 and x^3, respectively. The compatibility conditions lead to the differential-difference Darboux system,

$$\begin{aligned} \partial_1 \beta_{23} &= \beta_{21} T_2 \beta_{13}, \quad & \partial_1 \beta_{32} &= \beta_{31} T_3 \beta_{12}, \\ \triangle_2 \beta_{13} &= \beta_{12} \beta_{23}, \quad & \triangle_2 \beta_{31} &= \beta_{32} T_3 \beta_{21}, \\ \triangle_3 \beta_{12} &= \beta_{13} \beta_{32}, \quad & \triangle_3 \beta_{21} &= \beta_{23} T_2 \beta_{31}. \end{aligned}$$

Differential-difference case (two discrete variables)

Let us introduce a potential u via the relations

$$\triangle_2 u_1 = \beta_{12}\beta_{21}, \quad \triangle_3 u_1 = \beta_{13}\beta_{31}, \quad \triangle_2\triangle_3 u = -\ln(1-\beta_{23}\beta_{32}),$$

which are compatible modulo the Darboux system. Introducing the notation $m = \beta_{12}\beta_{23}\beta_{31}$ and $n = \beta_{13}\beta_{32}\beta_{21}$, one has

$$\triangle_2\triangle_3 u_1 = (m + n + \triangle_2 u_1 + \triangle_3 u_1)e^{\triangle_2\triangle_3 u} - \triangle_2 u_1 - \triangle_3 u_1,$$
$$mn = \triangle_2 u_1 \triangle_3 u_1 (1 - e^{-\triangle_2\triangle_3 u}).$$

Solving for m and n one obtains

$$m = \frac{b - \sqrt{b^2 - 4c}}{2}, \quad n = \frac{b + \sqrt{b^2 - 4c}}{2},$$

where

$$b = (\triangle_2\triangle_3 u_1 + \triangle_2 u_1 + \triangle_3 u_1)e^{-\triangle_2\triangle_3 u} - \triangle_2 u_1 - \triangle_3 u_1,$$
$$c = \triangle_2 u_1 \triangle_3 u_1 (1 - e^{-\triangle_2\triangle_3 u}).$$

Differential-difference case (two discrete variables)

Parametrizing rotation coefficients in the form

$$\beta_{12} = \sqrt{\triangle_2 u_1}\, e^{\varphi}, \quad \beta_{21} = \sqrt{\triangle_2 u_1}\, e^{-\varphi},$$
$$\beta_{13} = \sqrt{\triangle_3 u_1}\, e^{-\psi}, \quad \beta_{31} = \sqrt{\triangle_3 u_1}\, e^{\psi},$$
$$\beta_{23} = \sqrt{1 - e^{-\triangle_2 \triangle_3 u}}\, e^{\eta}, \quad \beta_{32} = \sqrt{1 - e^{-\triangle_2 \triangle_3 u}}\, e^{-\eta},$$

and substituting into the expressions for m, we obtain

$$\varphi + \psi + \eta = \ln \frac{m}{\sqrt{\triangle_2 u_1 \triangle_3 u_1 (1 - e^{-\triangle_2 \triangle_3 u})}}.$$

Under the same parametrization, the Darboux system takes the form

$$\triangle_3 \varphi = \ln \frac{\sqrt{\triangle_2 u_1}\sqrt{\triangle_2 \triangle_3 u_1 + \triangle_2 u_1}}{m + \triangle_2 u_1} - \triangle_2 \triangle_3 u, \quad \triangle_2 \psi = \ln \frac{\sqrt{\triangle_3 u_1}\sqrt{\triangle_2 \triangle_3 u_1 + \triangle_3 u_1}}{m + \triangle_3 u_1},$$

$$\partial_1 \eta = -\frac{1}{2} \frac{\triangle_2 \triangle_3 u_1}{e^{\triangle_2 \triangle_3 u} - 1} + \frac{\triangle_2 u_1 \triangle_3 u_1}{m} + \triangle_2 u_1.$$

Differential-difference case (two discrete variables)

Applying the operator $\partial_1 \triangle_2 \triangle_3$ to

$$\varphi + \psi + \eta = \ln \frac{m}{\sqrt{\triangle_2 u_1 \triangle_3 u_1 (1 - e^{-\triangle_2 \triangle_3 u})}}.$$

one obtains a differential-difference equation in terms of u,

$$\partial_1 \triangle_2 \left(\ln \frac{\sqrt{\triangle_2 u_1} \sqrt{\triangle_2 \triangle_3 u_1 + \triangle_2 u_1}}{m + \triangle_2 u_1} - \triangle_2 \triangle_3 u \right)$$

$$+ \partial_1 \triangle_3 \left(\ln \frac{\sqrt{\triangle_3 u_1} \sqrt{\triangle_2 \triangle_3 u_1 + \triangle_3 u_1}}{m + \triangle_3 u_1} \right)$$

$$+ \triangle_2 \triangle_3 \left(-\frac{1}{2} \frac{\triangle_2 \triangle_3 u_1}{e^{\triangle_2 \triangle_3 u} - 1} + \frac{\triangle_2 u_1 \triangle_3 u_1}{m} + \triangle_2 u_1 \right)$$

$$- \partial_1 \triangle_2 \triangle_3 \left(\ln \frac{m}{\sqrt{\triangle_2 u_1 \triangle_3 u_1 (1 - e^{-\triangle_2 \triangle_3 u})}} \right) = 0.$$

Differential-difference case (two discrete variables)

To reconstruct the corresponding Lagrangian, note that for a Lagrangian density $F = F(\triangle_2 u_1,\ \triangle_3 u_1,\ \triangle_2\triangle_3 u,\ \triangle_2\triangle_3 u_1)$, the corresponding Euler-Lagrange equation is

$$\partial_1\triangle_2\left(\frac{\partial F}{\partial(\triangle_2 u_1)}\right)+\partial_1\triangle_3\left(\frac{\partial F}{\partial(\triangle_3 u_1)}\right)+\triangle_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_2\triangle_3 u)}\right)$$
$$-\partial_1\triangle_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_2\triangle_3 u_1)}-\frac{\partial F}{\partial(\triangle_2 u_1)}-\frac{\partial F}{\partial(\triangle_3 u_1)}\right)=0.$$

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The Lagrangian density is

$$F=\Delta_2 u_1\ln\left(1+\frac{m}{\Delta_2 u_1}\right)+\Delta_3 u_1\ln\left(1+\frac{m}{\Delta_3 u_1}\right)$$

$$+\frac{1}{2}(\Delta_2 u_1+\Delta_3 u_1)\Delta_2\Delta_3 u+\Delta_2\Delta_3 u_1\ln(\Delta_2\Delta_3 u_1-m).$$

Discrete Case

We begin with a discrete linear system

$$\triangle_i H_j = \beta_{ij} H_i,$$

whose compatibility conditions lead to the discrete Darboux system,

$$\triangle_i \beta_{jk} = \beta_{ji} T_j \beta_{ik},$$

$i \neq j \neq k$. Here $\triangle_i = T_i - 1$ is the discrete x^i-derivative and T_i denotes unit shift in discrete variable x^i. One also has the relations

$$T_i \beta_{jk} = \frac{\beta_{ji}\beta_{ik} + \beta_{jk}}{1 - \beta_{ij}\beta_{ji}}$$

that follow from the Darboux system. In the discrete case, potential u is defined via the relations

$$\triangle_j \triangle_k u = -\ln(1 - \beta_{jk}\beta_{kj}),$$

which are compatible modulo the Darboux system.

Discrete Case

Introducing the notation $m = \beta_{12}\beta_{23}\beta_{31}$ and $n = \beta_{13}\beta_{32}\beta_{21}$, one has

$$mn = (1 - e^{-\triangle_1\triangle_2 u})(1 - e^{-\triangle_1\triangle_3 u})(1 - e^{-\triangle_2\triangle_3 u}),$$

$$\triangle_1\triangle_2\triangle_3 u = -\triangle_1\triangle_2 u - \triangle_1\triangle_3 u - \triangle_2\triangle_3 u$$

$$- \ln(e^{-\triangle_1\triangle_2 u} + e^{-\triangle_1\triangle_3 u} + e^{-\triangle_2\triangle_3 u} - m - n - 2).$$

Solving for m and n one obtains

$$m = \frac{b - \sqrt{b^2 - 4c}}{2}, \quad n = \frac{b + \sqrt{b^2 - 4c}}{2},$$

where

$$b = e^{-\triangle_1\triangle_2 u} + e^{-\triangle_1\triangle_3 u} + e^{-\triangle_2\triangle_3 u} - 2 - e^{-\triangle_1\triangle_2\triangle_3 u - \triangle_1\triangle_2 u - \triangle_1\triangle_3 u - \triangle_2\triangle_3 u},$$

$$c = (1 - e^{-\triangle_1\triangle_2 u})(1 - e^{-\triangle_1\triangle_3 u})(1 - e^{-\triangle_2\triangle_3 u}).$$

Parametrizing rotation coefficients in the form

$$\begin{array}{ll} \beta_{12} = \sqrt{1 - e^{-\triangle_1 \triangle_2 u}}\, e^{\varphi}, & \beta_{21} = \sqrt{1 - e^{-\triangle_1 \triangle_2 u}}\, e^{-\varphi}, \\ \beta_{13} = \sqrt{1 - e^{-\triangle_1 \triangle_3 u}}\, e^{-\psi}, & \beta_{31} = \sqrt{1 - e^{-\triangle_1 \triangle_3 u}}\, e^{\psi}, \\ \beta_{23} = \sqrt{1 - e^{-\triangle_2 \triangle_3 u}}\, e^{\eta}, & \beta_{32} = \sqrt{1 - e^{-\triangle_2 \triangle_3 u}}\, e^{-\eta}, \end{array}$$

and substituting into the expression for m, we we obtain

$$\varphi + \psi + \eta = \ln \frac{m}{\sqrt{(1 - e^{-\triangle_1 \triangle_2 u})(1 - e^{-\triangle_1 \triangle_3 u})(1 - e^{-\triangle_2 \triangle_3 u})}}.$$

Under the same parametrization, the Darboux system gives

$$\triangle_3 \varphi = -\ln\left(1 + \frac{m}{1 - e^{-\triangle_1 \triangle_2 u}}\right) - \frac{1}{2}\ln(1 - e^{-\triangle_1 \triangle_2 u})$$

$$+\frac{1}{2}\ln(1 - e^{-\triangle_1 \triangle_2 \triangle_3 u - \triangle_1 \triangle_2 u}) - \triangle_2 \triangle_3 u,$$

Discrete Case

$$\triangle_2\psi = -\ln\left(1+\frac{m}{1-e^{-\triangle_1\triangle_3 u}}\right)-\frac{1}{2}\ln(1-e^{-\triangle_1\triangle_3 u})$$

$$+\frac{1}{2}\ln(1-e^{-\triangle_1\triangle_2\triangle_3 u-\triangle_1\triangle_3 u})-\triangle_1\triangle_2 u,$$

$$\triangle_1\eta = -\ln\left(1+\frac{m}{1-e^{-\triangle_2\triangle_3 u}}\right)-\frac{1}{2}\ln(1-e^{-\triangle_2\triangle_3 u})$$

$$+\frac{1}{2}\ln(1-e^{-\triangle_1\triangle_2\triangle_3 u-\triangle_2\triangle_3 u})-\triangle_1\triangle_3 u.$$

Applying the operator $\triangle_1\triangle_2\triangle_3$ to

$$\varphi+\psi+\eta=\ln\frac{m}{\sqrt{(1-e^{-\triangle_1\triangle_2 u})(1-e^{-\triangle_1\triangle_3 u})(1-e^{-\triangle_2\triangle_3 u})}},$$

one obtains a difference equation in terms of u.

Discrete Case

To reconstruct the corresponding Lagrangian, note that for a Lagrangian density $F = F(\triangle_1\triangle_2 u,\ \triangle_1\triangle_3 u,\ \triangle_2\triangle_3 u,\ \triangle_1\triangle_2\triangle_3 u)$, the corresponding Euler-Lagrange equation is

$$\triangle_1\triangle_2\left(\frac{\partial F}{\partial(\triangle_1\triangle_2 u)}\right) + \triangle_1\triangle_3\left(\frac{\partial F}{\partial(\triangle_1\triangle_3 u)}\right) + \triangle_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_2\triangle_3 u)}\right)$$
$$-\triangle_1\triangle_2\triangle_3\left(\frac{\partial F}{\partial(\triangle_1\triangle_2\triangle_3 u)} - \frac{\partial F}{\partial(\triangle_1\triangle_2 u)} - \frac{\partial F}{\partial(\triangle_1\triangle_3 u)} - \frac{\partial F}{\partial(\triangle_2\triangle_3 u)}\right) = 0.$$

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Comparison gives the expressions for all first-order derivatives of F which, on integration, leads to the Lagrangian density

$$F=\mathrm{Li}_2\left(\frac{e^{-\triangle_1\triangle_2 u}}{1+m}\right)+\mathrm{Li}_2\left(\frac{e^{-\triangle_1\triangle_3 u}}{1+m}\right)+\mathrm{Li}_2\left(\frac{e^{-\triangle_2\triangle_3 u}}{1+m}\right)+\mathrm{Li}_2\left(\frac{(1+m)}{e^{\triangle_1\triangle_2\triangle_3 u}}\right)$$
$$-\mathrm{Li}_2\left(e^{-\triangle_1\triangle_2 u}\right)-\mathrm{Li}_2\left(e^{-\triangle_1\triangle_3 u}\right)-\mathrm{Li}_2\left(e^{-\triangle_2\triangle_3 u}\right)-\mathrm{Li}_2\left(\frac{1}{1+m}\right)$$
$$+(\triangle_1\triangle_2 u+\triangle_1\triangle_3 u+\triangle_2\triangle_3 u+\ln(1+m))\ln(1+m)$$
$$+\frac{1}{2}(\triangle_1\triangle_2 u\triangle_1\triangle_3 u+\triangle_1\triangle_2 u\triangle_2\triangle_3 u+\triangle_1\triangle_3 u\triangle_2\triangle_3 u).$$

Conclusion. Extended Darboux-KP Hierarchy

We expect that all 3D equations belonging to this hierarchy can be expressed via a sole function u, which depends on infinitely many independent variables

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Examples:

$$S = \int \left(u_{yy}^2 - u_{xx}u_{xt} + u_{xx}^2 u_{yy} + u_{xx}u_{xy}^2 + \frac{1}{4}u_{xx}^4 \right) dxdydt,$$

$$S = \int \left(u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3}u_{xx}^3 \right)^{3/2} dxdydt,$$

$$S = \int u_{xt}^{-2}(u_{xt}u_{yt} - u_{xx}u_{xt}^2)^{3/2} dxdydt.$$

Bogdanov–Ferapontov 3D Reductions

The theory of conjugate curvilinear coordinate nets is based on investigation of properties of the Darboux system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \quad i \neq j \neq k,$$

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$$H_i = c_i \partial_i^n \psi_i + \dots$$

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if $n = 1$, this is nothing but orthogonal curvilinear coordinate nets.