On linear deformations of the matrix product

Vladimir V. Sokolov

Higher School of Modern Mathematics MIPT,
Higher School of Economics

Moscow, Russia

vsokolov@landau.ac.ru

Yaroslavl', 25-th February, 2026

Joint research with I. Golubchik and A. Odesskii

References

1. I.Z. Golubchik and V.V. Sokolov *Factorization of the loop algebras and compatible Lie brackets*, Journal of Nonlinear Math. Phys., **12**(1), 343–350, 2005.

2. A.V. Odesskii and V. V. Sokolov, *Compatible Lie brackets related to elliptic curve*, J. of Math. Phys., 2006, **47**, 013506.

3. A.V. Odesskii and V. V. Sokolov, *Integrable matrix equations related to pairs of compatible associative algebras*, Journal Phys. A: Math. Gen., 2006, **39**, 12447–12456.

4. A.V. Odesskii and V. V. Sokolov, *Pairs of compatible associative algebras, classical Yang-Baxter equation and quiver representations*, Comm. in Math. Phys., 2008, **278**, no.**1**, 83–99.

Compatible Lie brackets.

Two Lie brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_1$ defined on the same finite dimensional vector space $\mathbf{V}$ are said to be compatible if

$$[\cdot,\cdot]_\lambda = [\cdot,\cdot] + \lambda[\cdot,\cdot]_1 \tag{1}$$

is a Lie bracket for any constant λ.

Suppose that the Lie algebra with the bracket $[\cdot,\cdot]$ is a semi-simple Lie algebra $\mathcal{G}$. Then there exists a formal series of the form

$$A_\lambda = \mathbf{1} + R\,\lambda + O(\lambda^2),$$

where the coefficient R is a constant linear operator on $\mathcal{G}$, such that

$$A_\lambda^{-1}\,[A_\lambda(X),\,A_\lambda(Y)] = [X,\,Y] + \lambda\,[X,\,Y]_1$$

is valid. It follows from this identity that

$$[X,\,Y]_1 = [R(X),\,Y] + [X,\,R(Y)] - R([X,\,Y]). \tag{2}$$

Lemma 1.

i) The operation $[\cdot,\cdot]_1$ is a Lie bracket iff there exists a linear operator $S:\mathcal{G}\to\mathcal{G}$ such that

$$R\big([R(X),Y]-[R(Y),X]\big)-[R(X),R(Y)]-R^2([X,Y])=$$

$$[S(X),Y]-[S(Y),X]-S([X,Y]).$$

ii) If this relation is satisfied, then the bracket $[\cdot,\cdot]_1$ is compatible with $[\cdot,\cdot]$.

In the special case $S=0$, the relation from Lemma 1 takes the form

$$R\big([R(X),Y]-[R(Y),X]\big)-[R(X),R(Y)]-R^2([X,Y])=0. \quad (3)$$

Proposition. Suppose that $R : \mathcal{G} \mapsto \mathcal{G}$ is a diagonalizable operator. Let $\lambda_1, ..., \lambda_k$ be the spectrum of R, and let $\mathcal{G}_i$ be the corresponding eigenspaces. The operator R satisfies (3) if and only if the subspaces $\mathcal{G}_i$, as well as $\mathcal{G}_i \oplus \mathcal{G}_j$ with distinct $i, j = 1, ..., k$, are Lie subalgebras of $\mathcal{G}_i$.

Proposition permits one to construct a k-parameter family of solutions

$$R = \sum_{i=1}^{k} \lambda_i \Pi_i,$$

where Π_i is the projection on $\mathcal{G}_i$. The parameters λ_i can be chosen arbitrarily.

In each specific case, the series A_λ is an operator-valued meromorphic function of the variable λ, which changes the basis such that the bracket (1) is reduced to the bracket $[\cdot,\cdot]$.

This function, if desired, can be regarded as a kind of Yang-Baxter r-matrix. Like solutions of the Yang-Baxter equation, A_λ-matrices are usually rational, trigonometric, or elliptic functions of λ. Examples of elliptic A_λ-matrices were found in [2].

Example. Suppose $\mathcal{G} = Mat_N$ with the bracket $[X, Y] = XY - YX$. Define $[\cdot,\cdot]_1$ by $[X, Y]_1 = XcY - YcX$, where c is a fixed matrix. Then (1) is a Lie bracket for any λ. In this case $A_\lambda : g \mapsto g + \lambda c g$.

Motivations.

1. Bi-Hamiltonian formalism. The formula

$$\{x_i,\, x_j\} = c_{ij}^k x_k, \qquad i, j = 1, \dots, N$$

defines a linear Poisson bracket iff c_{ij}^k are structural constants of a Lie algebra $\mathcal{G}$. Any linear deformation of this algebra defines a pencil of linear Poisson brackets. The Casimir function of the pencil defines a set of commuting Hamiltonians.

2. Integrable top-like systems. Consider the following system of ordinary differential equations for three vectors $u, v, w \in \mathbf{V}$:

$$w_t = [w,\, v] + w * w, \qquad v_t = [w,\, u] + w * v, \qquad u_t = w * u,$$

where

$$X * Y \stackrel{def}{=} [R(X),\, Y] - [X,\, R^*(Y)] + R^*([X, Y]),$$

and R^* denotes the operator adjoint to R with respect to the Killing form.

If $v = w = 0$ and $[Y, Y[_1 = XcY - YcX$, then $X * Y = cXY - YXc$ and the equation has the form

$$w_t = [w^2, c].$$

It turns out that the operators

$$\mathcal{L} = (A_\lambda^{-1})^*(\lambda u + v + \lambda^{-1} w), \qquad \mathcal{A} = \lambda^{-1} A_\lambda(w) \tag{4}$$

form a Lax pair for this system. As usual, the integrals of motion can be found from the expressions $\operatorname{tr} \mathcal{L}^k$, $k = 1, 2 \ldots$. Each of them depends on λ, which plays the role of a spectral parameter in the Lax pair, and therefore defines several first integrals.

3. Integrable hyperbolic systems. Consider a hyperbolic system of equations

$$u_x = [u,\, v], \qquad v_y = [v,\, u]_1, \tag{5}$$

where u and v belong to the vector space $\mathbf{V}$ endowed with a pair of Lie brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_1$. For the well-known integrable system of the principle chiral field

$$u_x = [u,\, v], \qquad v_y = [v,\, u]$$

the brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_1$ coincide with the matrix commutator.

Theorem 1. If the brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_1$ are compatible, then the system (5) has the Lax pair

$$\mathcal{L} = \frac{d}{dy} + \frac{1}{\lambda} A_\lambda(u), \qquad \mathcal{A} = \frac{d}{dx} + A_\lambda(v).$$

4. Factorization of the loop algebra over $\mathcal{G}$. Let $\mathfrak{g}$ be a Lie algebra. The Lie algebra $\mathfrak{g}((\lambda))$ of formal series of the form

$$\sum_{i=-n}^{\infty} g_i \lambda^i \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{Z}$$

is called the (extended) *loop algebra* over $\mathfrak{g}$.

Consider decompositions

$$\mathfrak{g}((\lambda)) = \mathfrak{g}[[\lambda]] \oplus \mathcal{U} \tag{6}$$

of the loop algebra into a direct sum of vector subspaces, the first of which is the Lie subalgebra $\mathfrak{g}[[\lambda]]$ of all Taylor series, and the second one is a Lie subalgebra. The Lie algebra $\mathcal{U}$ is called *factoring*, or *complementary*.

The simplest factoring subalgebra consists of polynomials in $\frac{1}{\lambda}$ with a zero free term:

$$\mathcal{U}^{st} = \Big\{ \sum_{i=1}^{n} g_i \lambda^{-i} \quad | \quad g_i \in \mathfrak{g}, \quad n \in \mathbb{N} \Big\}.$$

A factoring subalgebra $\mathcal{U}$ is called homogeneous if

$$\frac{1}{\lambda}\mathcal{U} \subset \mathcal{U}.$$

Proposition.

- i) Any homogeneous factoring subalgebra $\mathcal{U}$ has the form

$$\mathcal{U} = \Big\{ \sum_{i=1}^{k} \lambda^{-i} A_\lambda(g_i), \,|\, g_i \in \mathcal{G},\ k \in \mathbb{N} \Big\}, \tag{7}$$

where A_λ is a series of the form

$$A_\lambda = \mathbf{1} + R\,\lambda + O(\lambda^2).$$

- ii) The vector space (7) is a factoring Lie subalgebra iff A_λ satisfies the relation (3) for some Lie bracket $[\cdot,\cdot]_1$, compatible with the bracket on $\mathcal{G}$.

5. Solutions of the classical Yang–Baxter equation.

The operator A_λ is related to the Yang-Baxter equation. The operator form of the classical Yang-Baxter equation is given by

$$[r(u,w)x, r(u,v)y] = r(u,v)[r(v,w)x, y] + r(u,w)[x, r(w,v)y].$$

Here $r(u,v) \in End(\mathcal{G})$. The solution is called unitary if $< x, r(u,v)y >= - < r(v,u)x, y >$.

Theorem 2. *The operator*

$$r(u,v) = \frac{1}{u-v} A_u A_v^{-1}$$

satisfies the Yang-Baxter equation.

Remark. This solution is unitary with respect to an invariant form $(\cdot,\cdot)$ if the operator A_u is orthogonal.

Example. If $A_u = 1 + uR$, then

$$r(u,v) = \frac{1}{u-v} + (v+R)^{-1}.$$

Compatible associative algebras

Two associative algebras with multiplications $\star$ and $\circ$ defined on the same finite dimensional vector space $\mathbf{V}$ are said to be compatible if the multiplication

$$a \bullet b = a \star b + \lambda \, a \circ b$$

is associative for any constant λ.

Example 1. Let $\mathbf{V}$ be the vector space of polynomials of degree $\leqslant k-1$ in one variable, μ_1 and μ_2 be polynomials of degree k without common roots. Any polynomial Z, where $\deg Z \leqslant 2k-1$, can be uniquely represented in the form $Z = \mu_1 P + \mu_2 Q$, where $P, Q \in \mathbf{V}$. Define multiplications $\circ$ and $\star$ on $\mathbf{V}$ by the formula

$$X\,Y = \mu_1(X \circ Y) + \mu_2(X \star Y), \qquad X, Y \in \mathbf{V}.$$

It can be verified that any linear combination of these products is associative.

Assume that the associative algebra with multiplication $\star$ coincides with Mat_N. Since the matrix algebra is rigid, there exists a linear operator S_λ on Mat_n such that

$$S_\lambda(X)\, S_\lambda(Y) = S_\lambda\Big(XY + \lambda\ X \circ Y\Big). \tag{8}$$

If

$$S_\lambda = \mathbf{1} + R\ \lambda + O(\lambda^2), \tag{9}$$

then multiplication $\circ$ is given by

$$X \circ Y = R(X)Y + X R(Y) - R(XY). \tag{10}$$

where $R:\ Mat_n \to Mat_n$ is a linear operator.

Example 2. Let c be an element of $\mathbf{V}$ and $R : X \to cX$ be the operator of left multiplication by c . Then the corresponding multiplication $X \circ Y = X \star c \star Y$ is associative and compatible with $\star$.

Example 3. Suppose $a, b \in Mat_2$; then the product

$$X \circ Y = (aX - Xa)\,(bY - Yb)$$

is compatible with the standard product in Mat_2 . The corresponding operator R is given by

$$R(X) = a\,(Xb - bX).$$

In the case of Mat_2 any linear deformation of the matrix product is given by Examples 2,3.

If $a, b \in Mat_n,\ n > 2$ we need additional assumption $a^2 = b^2 = \mathbf{1}$.

The corresponding bi-Hamiltonian ODEs belong to the following class of matrix differential equations

$$\frac{dx}{dt} = [x,\, R(x) + R^*(x)],$$

where x is $n \times n$ matrix, R is a constant linear operator $R : Mat_n \to Mat_n$, and $*$ stands for the adjoint operator with respect to the bi-linear form trace (xy).

In the case of Example 2 we get

$$\frac{dx}{dt} = [x,\, xc + cx] = x^2\, c - c\, x^2.$$

for $n \times n$-matrix x and any constant matrix c.

Under reduction $x^T = -x$, $c^T = c$ the equation is a commuting flow for the n-dimensional Euler equation.

In the case of Example 3 we have

$$x_t = [x,\ bxa + axb + xba + bax], \tag{11}$$

where $a^2 = b^2 = \mathbf{1}$.

Equation (11) admits the following skew- symmetric reduction

$$x^T = -x, \qquad b = a^T.$$

Different integrable $so(n)$-models provided by this reduction are in one-to-one correspondence with equivalence classes with respect to the $SO(n)$ gauge action of $n \times n$ matrices a such that $a^2 = \mathbf{1}$.

In the case $n = 4$ the family (11) contains the Steklov and the Poincare integrable models on $so(4)$.

Corresponding algebraic structures

Any linear operator $R : Mat_n \to Mat_n$ can be written as

$$R(x) = a_1\, x\, b^1 + ... + a_{p+1}\, x\, b^{p+1},$$

where $a_i, b^i \in Mat_n$, with p being smallest possible. Then

$$R^*(x) = b^1\, x\, a_1 + ... + b^{p+1}\, x\, a_{p+1}.$$

For Example 2, $R(x) = cx$. In the case of Example 3 $R(x) = axb + bax$.

By the equivalence transformation

$$R \longrightarrow R + ad_\alpha + \beta\, \mathbf{1}$$

we reduce R to the form

$$R(x) = a_1\, x\, b^1 + ... + a_p\, x\, b^p + c\, x. \tag{12}$$

Let us study the structure of associative algebra $\mathcal{M}$ generated by a_i, b^i, c. Denote by $\mathcal{L}$ the vector space generated by these elements.

Lemma 1. We have

$$a_i a_j = \varphi_{i,j}^k a_k + \mu_{i,j}\mathbf{1}, \qquad b^i b^j = \psi_k^{i,j} b^k + \lambda^{i,j}\mathbf{1}$$

for some tensors $\varphi_{i,j}^k$, $\mu_{i,j}$, $\psi_k^{i,j}$, $\lambda^{i,j}$.

This means that the vector spaces spanned by $\mathbf{1}, a_1, \dots a_p$ and $\mathbf{1}, b^1, \dots b^p$ are associative algebras. We denote them by $\mathcal{A}$ and $\mathcal{B}$.

The algebras $\mathcal{A}$ and $\mathcal{B}$ have to be related by certain consistency conditions.

Lemma 2. For some tensor t_j^i the following relations hold:

$$b^i a_j = \psi_j^{k,i}\, a_k + \varphi_{j,k}^i\, b^k + t_j^i \mathbf{1} + \delta_j^i\, c.$$

$$b^i\, c = \lambda^{k,i} a_k - t_k^i\, b^k - \varphi_{k,l}^i \psi_s^{l,k}\, b^s - \varphi_{k,l}^i \lambda^{l,k} \mathbf{1},$$

$$c\, a_j = \mu_{j,k}\, b^k - t_j^k\, a_k - \varphi_{k,l}^s \psi_j^{l,k}\, a_s - \mu_{k,l} \psi_j^{l,k} \mathbf{1},$$

where

$$\varphi_{j,k}^s \psi_s^{l,i} = \varphi_{s,k}^l \psi_j^{s,i} + \varphi_{j,s}^i \psi_k^{l,s} + \delta_k^l t_j^i - \delta_j^i t_k^l - \delta_j^i \varphi_{s,r}^l \psi_k^{r,s},$$

$$\varphi_{j,k}^s t_s^i = \psi_j^{s,i} \mu_{s,k} + \varphi_{j,s}^i t_k^s - \delta_j^i \psi_k^{s,r} \mu_{r,s},$$

$$\psi_s^{k,i} t_j^s = \varphi_{j,s}^i \lambda^{k,s} + \psi_j^{s,i} t_s^k - \delta_j^i \varphi_{s,r}^k \lambda^{r,s}.$$

Associative bi-algebras

The simplest well-known example of such a structure can be described as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras with basis $A_1, \ldots, A_p$ and $B^1, \ldots, B^p$ and structural constants $\varphi^i_{j,k}$ and $\psi^{\alpha,\beta}_\gamma$: $\quad A_j A_k = \varphi^i_{j,k} A_i, \qquad B^\alpha B^\beta = \psi^{\alpha,\beta}_\gamma B^\gamma.$

Suppose that the structural constants satisfy the following identities:

$$\varphi^s_{j,k}\psi^{l,i}_s = \varphi^l_{s,k}\psi^{s,i}_j + \varphi^i_{j,s}\psi^{l,s}_k, \qquad 1 \leqslant i, j, k, l \leqslant p.$$

Then the bi-algebra $\mathcal{M}$ of dimension $2p + p^2$ with the basis $A_i, B^j, A_i B^j$ and relations

$$B^i A_j = \psi^{k,i}_j A_k + \varphi^i_{j,k} B^k$$

is associative. Note that $A_i B^j$ form an associative subalgebra of dimension p^2.

Invariant description. We have a non-degenerate scalar product $(\cdot,\cdot)$ on $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ such that

$$(a_1, a_2) = (b_1, b_2) = 0$$

and

$$(b_1 b_2, v) = (b_1, b_2 v), \qquad (v, a_1 a_2) = (v a_1, a_2)$$

for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$. We can choose a dual bases A_i and B_j such that $(A_i, B^j) = \delta_i^j$.

Invariant description

Definition. By week $\mathcal{M}$-structure on a linear space $\mathcal{L}$ we mean a collection of the following data:

- Two subspaces $\mathcal{A}$ and $\mathcal{B}$ and distinguished element $\mathbf{1} \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{L}$.
- Associative products $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with unity $\mathbf{1}$.
- Left action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ of the algebra $\mathcal{B}$ and right action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ of the algebra $\mathcal{A}$ on the space $\mathcal{L}$, which commute to each other.
- A non-degenerate symmetric scalar product $(\cdot, \cdot)$ on the space $\mathcal{L}$.

These data should satisfy the following properties:

1. $\dim \mathcal{A} \cap \mathcal{B} = \dim \mathcal{L}/(\mathcal{A} + \mathcal{B}) = \mathbf{1}$. Intersection of $\mathcal{A}$ and $\mathcal{B}$ is a one dimensional space spanned by the unity $\mathbf{1}$.

2. Restriction of the action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ to subspace $\mathcal{B} \subset \mathcal{L}$ is the product in $\mathcal{B}$. Restriction of the action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ to subspace $\mathcal{A} \subset \mathcal{L}$ is the product in $\mathcal{A}$.

3. $(a_1, a_2) = (b_1, b_2) = 0$ and

$$(b_1 b_2, v) = (b_1, b_2 v), \quad (v, a_1 a_2) = (v a_1, a_2)$$

for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$.

It follows from these properties that $(\cdot, \cdot)$ gives a non-degenerate pairing between $\mathcal{A}/\mathbb{C}1$ and $\mathcal{B}/\mathbb{C}1$, so $\dim \mathcal{A} = \dim \mathcal{B}$ and $\dim \mathcal{L} = 2 \dim \mathcal{A}$.

For given week $\mathcal{M}$-structure we can define an algebra $U(\mathcal{L})$ generated by $\mathcal{L}$.

Let us describe the structure of $U(\mathcal{L})$ explicitly. Let $\{\mathbf{1}, A_1, ..., A_p\}$ be a basis of $\mathcal{A}$ and $\{\mathbf{1}, B^1, ..., B^p\}$ be a dual basis of $\mathcal{B}$ (which means that $(A_i, B^j) = \delta_i^j$). Let $C \in \mathcal{L}$ doesn't belong to the sum of $\mathcal{A}$ and $\mathcal{B}$. Without loss of generality we may assume that $(\mathbf{1}, C) = 1$, $(C, C) = (C, A_i) = (C, B^j) = 0$. Such element C is uniquely determined by choosing basis in $\mathcal{A}$ and $\mathcal{B}$.

Proposition 1. The algebra $U(\mathcal{L})$ is defined by the following relations

$$A_i A_j = \varphi_{i,j}^k A_k + \mu_{i,j}\mathbf{1}, \qquad B^i B^j = \psi_k^{i,j} B^k + \lambda^{i,j}\mathbf{1}$$

$$B^i A_j = \psi_j^{k,i} A_k + \varphi_{j,k}^i B^k + t_j^i \mathbf{1} + \delta_j^i C,$$

$$B^i C = \lambda^{k,i} A_k + u_k^i B^k + p^i \mathbf{1}, \quad C A_j = \mu_{j,k} B^k + u_j^k A_k + q_i \mathbf{1}$$

for certain tensors $\varphi_{i,j}^k, \psi_k^{i,j}, \mu_{i,j}, \lambda^{i,j}, u_k^i, p^i, q_i$.

Let us define an element $K \in U(\mathcal{L})$ by the formula $K = A_i B^i + C$.

Definition. Week $\mathcal{M}$-structure on $\mathcal{L}$ is called $\mathcal{M}$-structure if $K \in U(\mathcal{L})$ is a central element of the algebra $U(\mathcal{L})$.

Proposition 2. For any $\mathcal{M}$-structure on $\mathcal{L}$ we have

$$p^i = -\varphi^i_{k,l}\lambda^{l,k}, \quad q_i = -\psi^{k,l}_i \mu_{l,k}, \quad u^j_i = -t^j_i - \varphi^j_{k,l}\psi^{l,k}_i.$$

Theorem 1. Any representation $U(\mathcal{L}) \to Mat_n$ given by

$$A_1 \to a_1, ..., A_p \to a_p, B^1 \to b^1, ..., B^p \to b^p, C \to c$$

defines associative product on Mat_n compatible with the usual product.

Example 4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are generated by elements $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A^{p+1} = B^{p+1} = \mathbf{1}$. Assume that $(B^i, A^{-i}) = \epsilon^i - 1$, $(\mathbf{1}, C) = 1$ and other scalar products are equal to zero. Here ϵ is a primitive root of unity of order p. Let

$$B^i A^j = \frac{\epsilon^{-j} - 1}{\epsilon^{-i-j} - 1} A^{i+j} + \frac{\epsilon^i - 1}{\epsilon^{i+j} - 1} B^{i+j}$$

for $i + j \neq 0$ modulo p and

$$B^i A^{-i} = 1 + (\epsilon^i - 1)C, \quad C A^i = \frac{1}{1 - \epsilon^i} A^i + \frac{1}{\epsilon^i - 1} B^i,$$

$$B^i C = \frac{1}{\epsilon^{-i} - 1} A^i + \frac{1}{1 - \epsilon^{-i}} B^i$$

for $i \neq 0$ modulo p. These formulas define an $\mathcal{M}$-structure.

The central element has the following form

$$K = C + \sum_{0<i<p} \frac{1}{\epsilon^i - 1} A^{-i} B^i.$$

Let a, t be linear operators in some vector space. Assume that $a^{p+1} = 1$, $at = \epsilon ta$ and the operator $t - 1$ is invertible. It is easy to check that the formulas

$$A \to a, \qquad B \to \frac{\epsilon t - 1}{t - 1} a, \qquad C \to \frac{t}{t - 1}$$

define a representation of the algebra $U(\mathcal{L})$.

Case of semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$

Suppose a vector space $\mathcal{L}$ is equipped with a weak $\mathcal{M}$-structure such that

$$\mathcal{A} = \oplus_{1\leqslant i\leqslant r} End(V_i), \qquad \mathcal{B} = \oplus_{1\leqslant j\leqslant s} End(W_j),$$

$$\dim V_i = m_i, \qquad \dim W_j = n_j.$$

Lemma. $\mathcal{L}$ as a right $\mathcal{A}$-module is isomorphic to $\oplus_{1\leqslant i\leqslant r}(V_i^\star)^{2m_i}$.

It is known that $\mathcal{L}$ as $\mathcal{A}\otimes\mathcal{B}$-module is isomorphic to

$$\oplus_{1\leqslant i\leqslant r, 1\leqslant j\leqslant s}(V_i^\star \otimes W_j)^{a_{i,j}},$$

for some $a_{i,j} \geqslant 0$.

Theorem. For any i, j

$$\sum_{j=1}^{s} a_{i,j} n_j = 2m_i, \qquad \sum_{i=1}^{r} a_{i,j} m_i = 2n_j. \tag{13}$$

The matrix of linear system (13) for (n, m) has the form

$$Q = \begin{pmatrix} 2 & -A \\ -A^t & 2 \end{pmatrix}.$$

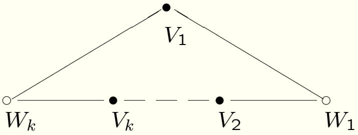
According to the result by E. Vinberg, if the kernel of indecomposable matrix Q contains an integer positive vector, them Q is the Cartan matrix of an affine Dynkin diagram.

Moreover, it follows from the structure of Q that this diagram is a simple laced affine Dynkin diagram with a partition of the set of vertices into two subsets such that vertices of the same subset are not connected.

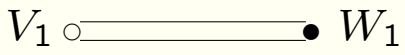
This is a complete list of such diagrams:

1. $A = (2)$. Here $r = s = 1$, $n_1 = m_1 = m$. The corresponding Dynkin diagram is of the type $\tilde{A}_1$.

2. $a_{i,i} = a_{i,i+1} = 1$ and $a_{i,j} = 0$ for other pairs i, j. Here $r = s = k \geq 2$, the indexes are taken modulo k, and $n_i = m_i = m$. The corresponding Dynkin diagram is $\tilde{A}_{2k-1}$.



$\tilde{A}_{2k-1}$

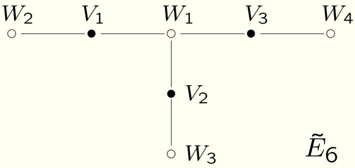


$\tilde{A}_1$

3. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Here $r = 3$, $s = 4$ and

$$n_1 = 3m,\ n_2 = n_3 = n_4 = m,$$
$$m_1 = m_2 = m_3 = 2m.$$

The Dynkin diagram is $\tilde{E}_6$:

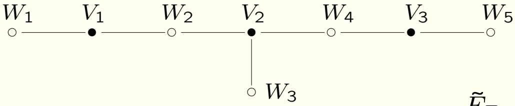


4. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$

Here $r = 3$, $s = 5$ and

$$n_1 = m, \;\; n_2 = 3m, \;\; n_3 = 2m, \;\; n_4 = 3m, \;\; n_5 = m,$$
$$m_1 = 2m, \quad m_2 = 4m, \quad m_3 = 2m.$$

The Dynkin diagram is $\tilde{E}_7$:



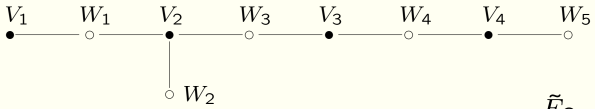
$\tilde{E}_7$

5. $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Here $r = 4, s = 5$,

$n_1 = 4m, \quad n_2 = 3m, \quad n_3 = 5m, \quad n_4 = 3m,$
$n_5 = m,$

$m_1 = 2m, \quad m_2 = 6m, \quad m_3 = 4m, \quad m_4 = 2m.$

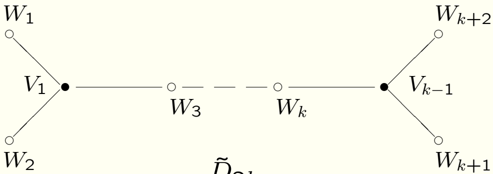
The Dynkin diagram is $\tilde{E}_8$:



$\tilde{E}_8$

6. $A = (1, 1, 1, 1)$. Here $r = 1, s = 4$ and
$n_1 = n_2 = n_3 = n_4 = m, \quad m_1 = 2m.$
The corresponding Dynkin diagram is $\tilde{D}_4$.

7. Here we have $r = k-1$, $s = k+2$ and $n_1 = n_2 = n_{k+1} = n_{k+2} = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_l = 2m$. The corresponding Dynkin diagram is $\tilde{D}_{2k}$, where $k \geq 3$.

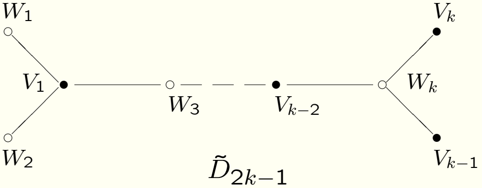


$\tilde{D}_{2k}$

8. $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{2,4} = a_{3,4} = a_{3,5} = \cdots = a_{k-2,k-1} = a_{k-2,k} = 1$, $a_{k-1,k} = a_{k,k} = 1$, and $a_{i,j} = 0$ for other (i,j).

Here we have $r = s = k \geq 3$, $n_1 = n_2 = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_{k-2} = 2m$, $m_{k-1} = m_k = m$. The corresponding Dynkin diagram is

$\tilde{D}_{2k-1}$:



Note that if $k = 3$, then $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{3,3} = 1$.

Summary.

Let $\mathcal{L}$ be a $\mathcal{M}$ structure with semisimple algebras $\mathcal{A}$ and $\mathcal{B}$. Then there is an affine Dynkin diagram of type A, D, or E such that:

1. There is a one-to-one correspondence between the set of vertices and the set of vector spaces $\{V_1, ..., V_r, W_1, ..., W_s\}$.

2. For any i, j the spaces V_i, V_j, as well as the spaces W_i, W_j, are not connected by edges.

3. The vector

$$(\dim V_1, ..., \dim V_r, \dim W_1, ..., \dim W_s)$$

equals J, where J is the minimal imaginary positive root of the Dynkin diagram.

Associative bi-algebras

The simplest example of such a structure can be described as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras with basis $A_1, \dots, A_p$ and $B^1, \dots, B^p$ and structural constants $\varphi^i_{j,k}$ and $\psi^{\alpha,\beta}_\gamma$. Suppose that the structural constants satisfy the following identities:

$$\varphi^s_{j,k}\psi^{l,i}_s = \varphi^l_{s,k}\psi^{s,i}_j + \varphi^i_{j,s}\psi^{l,s}_k, \qquad 1 \leqslant i,j,k,l \leqslant p.$$

Then the associative bi-algebra $\mathcal{M}$ of dimension $2p + p^2$ with the basis A_i, B^j, A_iB^j and relations

$$B^iA_j = \psi^{k,i}_j A_k + \varphi^i_{j,k}B^k$$

is associative. Note that $\mathcal{A}$ and $\mathcal{B}$ act on $\mathcal{M}$ from the right and from the left, correspondingly.