Applications of associative Yang-Baxter equation

for constructing integrable systems

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Plan of the talk:

- Integrable systems and R-matrices
- Associative Yang-Baxter equation
- Integrable tops, spin chains, Sklyanin algebras, Gaudin models, etc.
- Model of interacting tops
- R-matrix valued Lax pairs and its relation to quantum interacting tops
- Long-range spin chains
- BC-version of associative Yang-Baxter equation and K-matrices

Integrable systems in classical mechanics

Hamiltonian mechanics: Hamiltonian function H and Poisson brackets $\{\ ,\ \}$ provide equations of motion $\dot{f} = \{H, f\}$.

Many-body systems: N particles with positions $q_i \in \mathbb{C}$ and momenta $p_i \in \mathbb{C}$ and Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j}^{N} U(q_i - q_j)\,.$$

Canonical Poisson structure:

$$\{p_i, q_j\} = \delta_{ij}\,, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0\,, \quad i, j = 1...N\,.$$

We need: N independent integrals of motion $H_k(p, q)$ with the involution property

$$\{H_i, H_j\} = 0\,.$$

Integrable Euler-Arnold tops

Dynamical variables:

$$S = \sum_{i,j=1}^{N} S_{ij} E_{ij} \in \mathrm{Mat}(N)\,,$$

where E_{ij} – is the standard basis in $\mathrm{Mat}(N, \mathbb{C})$.
Euler equation:

$$\dot{S} = [S, J(S)]\,,$$

where

$$J(S) = \sum J_{ij,kl} S_{kl} E_{ij}$$

is a linear functional and $J_{ij,kl}$ – set of constants. From viewpoint of mechanics J is a multidimensional analogue of the inverse inertia tensor.

Hamiltonian:

$$H = \frac{1}{2}\,\mathrm{tr}(SJ(S))$$

Poisson brackets are given by Poisson-Lie structure on gl_N^*:

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}\,.$$

Equations of motion are represented in the Lax form with spectral parameter z:

$$\dot{L}(z) = [L(z), M(z)]\,, \quad \forall z \quad \dot{L}(z) = \{H, L(z)\} = \sum_{ij} E_{ij}\{H, L_{ij}(z)\}$$

Then $H_k(z) = \mathrm{tr}(L^k(z))$ – are generating functions for integrals of motion:

$$\mathrm{tr}(L^k(z)) = \sum_m (z - z_0)^m H_{k,m}\,, \qquad \frac{d}{dt} H_{k,m} = 0 \;\forall k, m$$

Define $\{L_1(z)\,, L_2(w)\} = \sum_{ijkl} E_{ij} \otimes E_{kl}\,\{L_{ij}(z), L_{kl}(w)\}$. If there exists $r_{12} \in \mathrm{Mat}^{\otimes 2}$ (classical r-matrix) satisfying relation

$$\{L_1(z)\,, L_2(w)\} = [L_1(z), r_{12}(z,w)] - [L_2(w), r_{21}(w,z)]\,,$$

$$r_{12}(z,w) = \sum_{ijkl} r_{ij,kl}(z,w)\, E_{ij} \otimes E_{kl}\,, \qquad r_{21}(w,z) = \sum_{ijkl} r_{ij,kl}(w,z)\, E_{kl} \otimes E_{ij}\,,$$

where $L_1(z) = L(z) \otimes 1$, $L_2(w) = 1 \otimes L(w)$, then $\{H_{i,m}, H_{j,n}\} = 0$.

Jacobi identity in $\mathrm{Mat}^{\otimes 3}$ for the Poisson brackets

$$\{\{L_1(z_1), L_2(z_2)\}\, L_3(z_3)\} + \text{cycl.} = 0\,, \quad L_3(z_3) = 1_N \otimes 1_N \otimes L(z_3)$$

is fulfilled if the classical Yang-Baxter equation holds true

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\,, \quad r_{ij} = r_{ij}(z_i - z_j)\,, \quad r_{12}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes E_{kl} \otimes 1_N\,,$$

$$r_{23}(z) = \sum_{ijkl} r_{ij,kl}(z)\, 1_N \otimes E_{ij} \otimes E_{kl}\,, \quad r_{13}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes 1_N \otimes E_{kl}\,.$$

In quantum case it is generalized to the quantum Yang-Baxter equation for quantum R-matrix:

$$R^\hbar_{12}\, R^\hbar_{13}\, R^\hbar_{23} = R^\hbar_{23}\, R^\hbar_{13}\, R^\hbar_{12}\,, \qquad R^\hbar_{ij} = R^\hbar_{ij}(z_i - z_j)$$

In the quasi-classical limit

$$R^\hbar_{12} = 1 \otimes 1 + \hbar\, r_{12} + O(\hbar^2)$$

the classical YB equation is reproduced.

The Calogero-Moser model:

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i<j}^{N} \wp(q_i - q_j)\,, \qquad \wp(x) \to \frac{1}{\sin^2(x)} \to \frac{1}{x^2}$$

where ν – coupling constant, $\wp(q)$ – Weierstrass $\wp$-function. Its equations of motion are written in the Lax form with

$$L(z) = \begin{pmatrix} p_1 & \nu\phi(z, q_1 - q_2) & \nu\phi(z, q_1 - q_3) & \dots & \nu\phi(z, q_1 - q_N) \\ \nu\phi(z, q_2 - q_1) & p_2 & \nu\phi(z, q_2 - q_3) & \dots & \nu\phi(z, q_2 - q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu\phi(z, q_N - q_1) & \nu\phi(z, q_N - q_2) & \nu\phi(z, q_N - q_3) & \dots & p_N \end{pmatrix}$$

where z – spectral parameter (Krichever 1980). It is a local coordinate on (elliptic) curve.

$$L_{ij}(z) = \delta_{ij} p_i + \nu(1-\delta_{ij})\,\phi(z, q_{ij})\,, \quad q_{ij} = q_i - q_j\,, \quad \phi(z,q) = \frac{\vartheta'(0)\vartheta(q+z)}{\vartheta(q)\,\vartheta(z)}\,,$$

$$M_{ij}(z) = \nu d_i \delta_{ij} + \nu(1-\delta_{ij}) f(z, q_{ij})\,, \quad d_i = -\sum_{k\neq i}^{N} f(0, q_{ik})\,, \quad f(z,q) = \partial_q \phi(z,q)\,.$$

Here $\phi(z,q)$ – elliptic Kronecker function:

$$\phi(\eta, z) = \frac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)} \;\to\; \coth(\eta) + \coth(z) \;\to\; 1/\eta + 1/z$$

It satisfies the genus 1 Fay identity (addition formula):

$$\phi(\hbar, q_1 - q_2)\phi(\eta, q_2 - q_3) = \phi(\hbar - \eta, q_1 - q_2)\phi(\eta, q_1 - q_3) + \phi(\eta - \hbar, q_2 - q_3)\phi(\hbar, q_1 - q_3)$$

Degeneration ($\hbar = \eta = z$):

$$\phi(z, q_{ab}) f(z, q_{bc}) - f(z, q_{ab})\phi(z, q_{bc}) = \phi(z, q_{ac})(\wp(q_{ab}) - \wp(q_{bc}))\,, \quad q_{ab} = q_a - q_b$$

$$f(z,q) = \partial_q \phi(z,q)\,, \qquad f(0,q) = -\wp(q) + const$$

These identities are widely used in integrable systems (Lax equations, R-matrix structures,...)

Quantum Yang-Baxter equation:

$$R^{\hbar}_{12}(q_1,q_2)R^{\hbar}_{13}(q_1,q_3)R^{\hbar}_{23}(q_2,q_3)=R^{\hbar}_{23}(q_2,q_3)R^{\hbar}_{13}(q_1,q_3)R^{\hbar}_{12}(q_1,q_2)$$

Below we use notations $R^{\hbar}_{12}(q_1,q_2)=R^{\hbar}_{12}(q_1-q_2)\equiv R^{\hbar}_{12}(q_{12})\equiv R^{\hbar}_{12}$

Associative Yang-Baxter equation::

$$R^{\hbar}_{12}(q_1-q_2)R^{\eta}_{23}(q_2-q_3)=R^{\eta}_{13}(q_1-q_3)R^{\hbar-\eta}_{12}(q_1-q_2)+R^{\eta-\hbar}_{23}(q_2-q_3)R^{\hbar}_{13}(q_1-q_3)$$

S. Fomin, An. Kirillov M. Aguiar, A. Polishchuk
Example of common solution: rational Yang's R-matrix

$$R^{\rm Yang}_{12}(q_1,q_2)=\frac{1\otimes 1}{\hbar}+\frac{P_{12}}{q_1-q_2}\,,\quad P_{12}=\sum_{i,j=1}^{N}E_{ij}\otimes E_{ji}\,,$$

where P_{12} – matrix permutation operator: $P_{12}(a\otimes b)=b\otimes a$ for $a,b\in\mathbb{C}^N$.

...,trigonometric R-matrices (XXZ),..., elliptic Baxter-Belavin R-matrix (XYZ).
Skew-symmetric ($R^{\hbar}_{12}(z)=-R^{-\hbar}_{21}(-z)$) and unitary ($R^{\hbar}_{12}(z)R^{\hbar}_{21}(-z)\sim 1\otimes 1$) solution of AYBE is also solution to QYB.

R-matrix is a matrix generalization of the Kronecker function ϕ

In the scalar case the QYB is an empty condition while the associative Yang-Baxter equation

$$R^{\hbar}_{12}(q_{12})R^{\eta}_{23}(q_{23}) = R^{\eta}_{13}(q_{13})R^{\hbar-\eta}_{12}(q_{12}) + R^{\eta-\hbar}_{23}(q_{23})R^{\hbar}_{13}(q_{13})$$

turns into the addition formula (considered as a functional equation for ϕ)

$$\phi(\hbar, q_{12})\phi(\eta, q_{23}) = \phi(\hbar-\eta, q_{12})\phi(\eta, q_{13}) + \phi(\eta-\hbar, q_{23})\phi(\hbar, q_{13}), \quad q_{ij} = q_i - q_j$$

One more example. Scalar identity

$$(E_1(q_{12}) + E_1(q_{23}) + E_1(q_{31}))^2 = \wp(q_{12}) + \wp(q_{23}) + \wp(q_{31}), \quad E_1(x) = \partial_x \log\vartheta(x)$$

is generalized to

$$\Big(r_{12}(q_{12}) + r_{23}(q_{23}) + r_{31}(q_{31})\Big)^2 = \mathrm{Id}\Big(\wp(q_{12}) + \wp(q_{23}) + \wp(q_{31})\Big)$$

if $R^{\hbar}_{12}(q)$ satisfies the associative Yang-Baxter equation and

$$R^{\hbar}_{12}(q) = \frac{1}{\hbar}\, 1\otimes 1 + r_{12}(q) + \hbar\, m_{12}(q) + O(\hbar^2).$$

R-matrices satifying AYBE – matrix generalizations of elliptic functions

$$\phi(\hbar, q_{ab}) f(\hbar, q_{bc}) - f(\hbar, q_{ab}) \phi(\hbar, q_{bc}) = \phi(\hbar, q_{ac})(\wp(q_{ab}) - \wp(q_{bc}))\,, \quad f(\hbar, x) = \partial_x \phi(\hbar, x)$$

$$R^{\hbar}_{ab} F^{\hbar}_{bc} - F^{\hbar}_{ab} R^{\hbar}_{bc} = F^{0}_{bc} R^{\hbar}_{ac} - R^{\hbar}_{ac} F^{0}_{ab}\,, \quad F^{\hbar}_{ab}(x) = \partial_x R^{\hbar}_{ab}(x)$$

Local expansion near $\hbar = 0$:

$$\phi(\hbar, q) = \hbar^{-1} + E_1(q) + \hbar\,(E_1^2(q) - \wp(q))/2 + O(\hbar^2)\,, \quad E_1(q) = \vartheta'(q)/\vartheta(q)$$

The quasi-classical limit

$$R^{\hbar}_{12}(q) = \frac{1}{\hbar}\, 1 \otimes 1 + r_{12}(q) + \hbar\, m_{12}(q) + O(\hbar^2)\,.$$

where $r_{12}(q)$ – classical r-matrix and

$$m_{12}(q) = \frac{1}{2}\, \left(r^2_{12}(q) - 1 \otimes 1\, \wp(q)\right)\,.$$

$R^{\hbar}_{12}(q)$ – matrix analogue for the function $\phi(\hbar, q)$, and $r_{12}(q)$ – matrix analogue for the function $E_1(q)$ ($E_1(x) = \partial_x \log \vartheta(x)$).

Integrable Euler-Arnold tops

Dynamical variables:

$$S = \sum_{i,j=1}^{N} S_{ij} E_{ij} \in \mathrm{Mat}(N)\,,$$

where E_{ij} – is the standard basis in $\mathrm{Mat}(N, \mathbb{C})$.
Euler equation:

$$\dot{S} = [S, J(S)]\,,$$

where

$$J(S) = \sum J_{ij,kl} S_{kl} E_{ij}$$

is a linear functional and $J_{ij,kl}$ – set of constants. From viewpoint of mechanics J is a multidimensional analogue of the inverse inertia tensor.

Hamiltonian:

$$H = \frac{1}{2}\,\mathrm{tr}(SJ(S))$$

Poisson brackets are given by Poisson-Lie structure on gl_N^*:

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}\,.$$

Let R be a solution to the associative Yang-Baxter equation with the properties of unitarity, skew-symmetry and the quasi-classical expansion

$$R^{\hbar}_{12}(z) = \frac{1}{\hbar}\, 1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2)\,.$$

Then the Lax equation

$$\dot{L}(z,S) = [L(z,S), M(z,S)]$$

holds true identically in z for the Lax pair:

$$L(z,S) = \mathrm{tr}_2(r_{12}(z)S_2)\,, \quad M(z,S) = \mathrm{tr}_2(m_{12}(z)S_2)\,, \quad S_2 = 1 \otimes S$$

$$r_{12}(z) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij} \otimes E_{kl} \quad \mathrm{tr}_2(r_{12}(z)S_2) = \sum_{ijkl} r_{ij,kl}(z)\, E_{ij}\, \mathrm{tr}(E_{kl}S) = \sum_{ijkl} r_{ij,kl}(z)\, S_{lk}\, E_{ij}\,,$$

The Lax equation itself is equivalent to the Euler equation with

$$J(S) = \mathrm{tr}_2(m_{12}(0)S_2)\,.$$

<u>We obtained a family of classical integrable tops</u> using data of a solution to AYBE.

Example: 7-vertex trigonometric R-matrix

$$R^{\hbar}(z) = \begin{pmatrix} \coth(z)+\coth(\hbar) & 0 & 0 & 0 \\ 0 & \sinh^{-1}(\hbar) & \sinh^{-1}(z) & 0 \\ 0 & \sinh^{-1}(z) & \sinh^{-1}(\hbar) & 0 \\ -4\,e^{-2\Lambda}\sinh(z+\hbar) & 0 & 0 & \coth(z)+\coth(\hbar) \end{pmatrix}$$

provides

$$J(S) = \frac{1}{6}\begin{pmatrix} 2S_{11}-S_{22} & 0 \\ -24\,e^{-2\Lambda}S_{12} & -S_{11}+2S_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Underlying identities. Degenerations in ”Planck constants” $\hbar$ and η. Beginning from AYBE

$$R_{12}^{\hbar}R_{23}^{\eta}=R_{13}^{\eta}R_{12}^{\hbar-\eta}+R_{23}^{\eta-\hbar}R_{13}^{\hbar}\,,\qquad R_{ab}^{\hbar}=R_{ab}(z_a-z_b)$$

in the limit $\eta\to\hbar$ it gives

$$R_{12}^{\hbar}R_{23}^{\hbar}=R_{13}^{\hbar}r_{12}+r_{23}R_{13}^{\hbar}-\partial_{\hbar}R_{13}^{\hbar}\,.$$

By changing indices $1\leftrightarrow 3$ (i.e. conjugating equation by P_{13} and renaming $z_1\leftrightarrow z_3$), changing also $\hbar\to-\hbar$ and then using skew-symmetry $R_{ab}^{\hbar}(z)=-R_{ba}^{-\hbar}(-z)$ it transforms into

$$R_{23}^{\hbar}R_{12}^{\hbar}=R_{13}^{\hbar}r_{23}+r_{12}R_{13}^{\hbar}-\partial_{\hbar}R_{13}^{\hbar}\,.$$

Subtracting these equations gives

$$[R_{12}^{\hbar},R_{23}^{\hbar}]=[R_{13}^{\hbar},r_{12}]-[R_{13}^{\hbar},r_{23}]\,.$$

Taking the limit $\hbar\to 0$ and using the classical Yang-Baxter equation

$$[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0\,,\quad r_{ab}=r_{ab}(z_a-z_b)$$

results

$$[r_{12},m_{13}+m_{23}]=[r_{23},m_{12}+m_{13}]\,,\quad m_{ab}=m_{ab}(z_a-z_b)$$

Underlying identities. Degenerations in spectral parameters z_1, z_2, z_3.

Write down degenerated AYBE with $z_3 = 0$:

$$[R^{\eta}_{13}(z_1), r_{12}(z_1 - z_2)] = [R^{\eta}_{12}(z_1 - z_2), R^{\eta}_{23}(z_2)] + [R^{\eta}_{13}(z_1), r_{23}(z_2)]$$

and consider the limit $z_2 \to 0$ (together with renaming $z_1 := z$). The simple pole at $z_2 = 0$ cancel out due to $[R^{\eta}_{12}(z), P_{23}] + [R^{\eta}_{13}(z), P_{23}] = 0$ by definition of permutation operator. Finally,

$$[R^{\eta}_{13}(z), r_{12}(z)] = [R^{\eta}_{12}(z), R^{\eta,(0)}_{23}] + [R^{\eta}_{13}(z), r^{(0)}_{23}] - [\partial_z R^{\eta}_{12}(z), P_{23}]$$

Here we use expansions near $\hbar = 0$ (the classical limit) and near $z = 0$:

$$R^{\hbar}_{12}(z) = \frac{1}{\hbar} 1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2)$$

$$R^{\hbar}_{12}(z) = \frac{1}{z} P_{12} + R^{\hbar,(0)}_{12} + z R^{\hbar,(1)}_{12} + O(z^2)\,,$$

$$R^{\hbar,(0)}_{12} = \frac{1}{\hbar} 1_N \otimes 1_N + r^{(0)}_{12} + O(\hbar)\,, \qquad r_{12}(z) = \frac{1}{z} P_{12} + r^{(0)}_{12} + O(z)\,.$$

$P_{12} = \sum_{i,j} E_{ij} \otimes E_{ji}$ – permutation operator: $P_{12}(u \otimes v) = v \otimes u$ for $u, v \in \mathbb{C}^N$.

Relativistic tops

The Lax equation

$$\dot{L}(z,S) = [L(z,S), M(z,S)]$$

is equivalent to the Euler's equations

$$\dot{S} = [S, J(S)]$$

for the Lax pair

$$L^\eta(z,S) = \mathrm{tr}_2(R^\eta_{12}(z)S_2)\,, \qquad M^\eta(z,S) = -\mathrm{tr}_2(r_{12}(z)S_2)$$

with the following inverse inertia tensor:

$$J^\eta(S) = \mathrm{tr}_2\Big((R^{\eta,(0)}_{12} - r^{(0)}_{12})S_2\Big)\,,$$

where the coefficients of the expansions near $z = 0$ are used.

$$R^\hbar_{12}(z) = \frac{1}{z}\,P_{12} + R^{\hbar,(0)}_{12} + zR^{\hbar,(1)}_{12} + O(z^2)\,,$$

$$R^{\hbar,(0)}_{12} = \frac{1}{\hbar}\,1_N \otimes 1_N + r^{(0)}_{12} + O(\hbar)\,, \qquad r_{12}(z) = \frac{1}{z}\,P_{12} + r^{(0)}_{12} + O(z)\,.$$

Hamiltonian structure for the relativistic top: Sklyanin type algebras

The quadratic r-matrix structure

$$\{L_1^\eta(z,S), L_2^\eta(w,S)\} = [L_1^\eta(z,S)L_2^\eta(w,S), r_{12}(z-w)]\,,$$

leads to the following quadratic Poisson brackets

$$\{S_1, S_2\} = [S_1 S_2, r_{12}^{(0)}] + [E^\eta(S)_1 S_2, P_{12}]\,, \qquad E^\eta(S)_1 = \mathrm{tr}_3(R_{13}^{\eta,(0)} S_3)\,.$$

for the defined above Lax matrices. Being written in components it takes the form:

$$\{S_{ij}, S_{kl}\} = (E_{il}^\eta S_{kj} - E_{kj}^\eta S_{il}) + \sum_{a,b=1}^{N} (S_{ia} S_{kb} r_{aj,bl}^{(0)} - r_{ia,kb}^{(0)} S_{aj} S_{bl})\,,$$

where

$$E^\eta(S)_{ij} = \sum_{m,n=1}^{N} R_{ij,mn}^{\eta,(0)} S_{nm}\,.$$

In the elliptic case these are the brackets for the GL_N version of Sklyanin algebra. In the relativistic case the Hamiltonian is given by

$$H^{\mathrm{rel}} = \mathrm{tr}(S)\,.$$

Integrable chains. Consider a set of $\mathcal{L}^\eta(z,S^a)$, $a=1,...,n$ attached to n sites. Then the monodromy matrix

$$T(z)=\mathcal{L}(z,S^1)\mathcal{L}(z,S^2)...\mathcal{L}(z,S^n)$$

also satisfies

$$\{T_1(z),T_2(w)\}=[T_1(z)T_2(w),r_{12}(z-w)]\,,$$

and $\mathrm{tr}(T(z))$ is a generating function of commuting Hamiltonians.
In the **special case**

$$S^a=\xi^a\otimes\psi^a\,,\qquad (\psi^a,\xi^a)=c=\mathrm{const}\,,$$

where ξ^a are N-dimensional column-vectors, and ψ^a are N-dimensional row-vectors, and (ψ^a,ξ^a) is their scalar product. Then the Lax pair

$$L^k(z)=\mathcal{L}(z,S^k)=\mathrm{tr}_2\Big(R^\eta_{12}(z)S^k_2\Big)\,,\quad M^k(z)=-\mathrm{tr}_2\Big(r_{12}(z)S^{k+1,k}_2\Big)\,,\quad S^{k+1,k}=\frac{\xi^{k+1}\otimes\psi^k}{(\psi^k,\xi^{k+1})}$$

satisfies the discrete Zakharov-Shabat equation: $\dot{L}^k(z)-L^k(z)M^k(z)+M^{k-1}(z)L^k(z)=0$. It holds true identically in spectral parameter z and provides the following equations of motion:

$$\dot{S}^k=E^0(S^{k,k-1})S^k-S^kE^0(S^{k+1,k})+S^{k,k-1}E^\eta(S^k)-E^\eta(S^k)S^{k+1,k}$$

with $E^\eta(S)=\mathrm{tr}_3(R^{\eta,(0)}_{13}S_3)$.

Limiting cases of integrable chains: Gaudin model.

Consider the inhomogeneous chain

$$T(z)=\mathcal{L}(z-z_1,S^1)\mathcal{L}(z-z_2,S^2)...\mathcal{L}(z-z_n,S^n)$$

and the non-relativistic limit $\eta\to 0$. Then one gets the classical Gaudin model. It is described by the Lax matrix

$$L^{\mathrm{G}}(z)=\sum_{a=1}^{n}\mathrm{tr}_2\Big(r_{12}(z-z_a)S_2^a\Big)\,.$$

The Poisson brackets become Poisson-Lie linear brackets on $\mathrm{gl}_N^{*\times n}$: $\{S_1^a,S_2^b\}=\delta^{ab}[P_{12},S_1^a]$.. The set of Hamiltonians:

$$H_a^{\mathrm{G}}=\sum_{c:c\neq a}^{n}\mathrm{tr}_{1,2}\Big(r_{12}(z_a-z_b)S_1^aS_2^b\Big)\,,\quad H_0^{\mathrm{G}}=\sum_{b\neq c}^{n}\mathrm{tr}_{1,2}\Big(m_{12}(z_b-z_c)S_1^bS_2^c\Big)\,,$$

where m_{12} is from $R_{12}^{\hbar}(z)=\hbar^{-1}1_N\otimes 1_N+r_{12}(z)+\hbar\, m_{12}(z)+O(\hbar^2)$. The Poisson commutativity

$$\{H_a^{\mathrm{G}},H_b^{\mathrm{G}}\}=0$$

is due to the classical Yang-Baxter equation, while

$$\{H_a^{\mathrm{G}},H_0^{\mathrm{G}}\}=0$$

comes from identity (derived from AYBE)

$$[r_{12},m_{13}+m_{23}]=[r_{23},m_{12}+m_{13}]\,,\quad m_{ab}=m_{ab}(z_a-z_b)\,.$$

Quantum Gaudin model.

Quantization of the classical Hamiltonians of Hamiltonians:

$$H_a^{\mathrm{G}} = \sum_{c:c\neq a}^{n} \mathrm{tr}_{1,2}\Big(r_{12}(z_a - z_b) S_1^a S_2^b\Big)\,,$$

$$H_0^{\mathrm{G}} = \sum_{b\neq c}^{n} \mathrm{tr}_{1,2}\Big(m_{12}(z_b - z_c) S_1^b S_2^c\Big)\,.$$

in the fundamental representation of gl_N Lie algebra is given by operators

$$\hat{H}_a^{\mathrm{G}} = \sum_{c:c\neq a}^{n} r_{ab}(z_a - z_b) \in \mathrm{Mat}(N,\mathbb{C})_N^{\otimes n}\,,$$

and

$$\hat{H}_0^{\mathrm{G}} = \sum_{b\neq c}^{n} m_{bc}(z_b - z_c) \in \mathrm{Mat}(N,\mathbb{C})_N^{\otimes n}\,.$$

Commutativity

$$[\hat{H}_a^{\mathrm{G}}, \hat{H}_b^{\mathrm{G}}] = 0\,, \qquad [\hat{H}_a^{\mathrm{G}}, \hat{H}_0^{\mathrm{G}}] = 0$$

again follows from the above mentioned identities.

Heat equation and KZ equations. The Kronecker function $\phi(z,u|\tau)$ satisfies **the heat equation**

$$2\pi\imath\partial_\tau\phi(z,u|\tau)=\partial_z\partial_u\phi(z,u|\tau)\,,$$

which follows from the heat equation for theta-function $4\pi\imath\partial_\tau\vartheta(z|\tau)=\partial_z^2\vartheta(z|\tau)\,.$
Suppose R-matrix solving AYBE depends on some parameter τ and satisfies also the heat equation

$$2\pi\imath\partial_\tau R_{12}^\hbar(z)=\partial_z\partial_\hbar R_{12}^\hbar(z)\,.$$

This is true for the properly normalized Baxter-Belavin elliptic R-matrix (and certain degenerations). Then in the limit $\hbar\to 0$ we get

$$2\pi\imath\partial_\tau r_{12}(z)=\partial_z m_{12}(z)\,.$$

This allows to define commuting Knizhnik-Zamolodchikov connections:

$$\nabla_a=\partial_{z_a}+\hat{H}_a^{\mathrm{G}}=\partial_{z_a}+\sum_{c:c\neq a}^n r_{ab}(z_a-z_b)\,,$$

$$\nabla_\tau=2\pi\imath\partial_\tau+\hat{H}_0^{\mathrm{G}}=2\pi\imath\partial_\tau+\sum_{b\neq c}^n m_{bc}(z_b-z_c)$$

and

$$[\nabla_a,\nabla_b]=0\,,\qquad [\nabla_a,\nabla_\tau]=0\,.$$

Limiting cases of integrable chains: 1+1 Landau-Lifshitz model.
Return back to the chain

$$\dot{L}^k(z) - L^k(z)M^k(z) + M^{k-1}(z)L^k(z) = 0\,.$$

$$\dot{S}^k = E^0(S^{k,k-1})S^k - S^k E^0(S^{k+1,k}) + S^{k,k-1}E^\eta(S^k) - E^\eta(S^k)S^{k+1,k}$$

with $E^\eta(S) = \mathrm{tr}_3(R_{13}^{\eta,(0)}S_3)$.
In the continuous limit we get higher rank Landau-Lifshitz equation. We again deal with the rank one case (below x is a coordinate on a circle)

$$S(x) = \xi(x)\otimes\eta(x) \in \mathrm{Mat}(N,\mathbb{C})\,, \qquad (\eta(x),\xi(x)) = c = \mathrm{const}\,.$$

Then the equations of motion take the form

$$\partial_t S = \frac{1}{c}\,[S,\partial_x^2 S] + \frac{2c}{N}\,[S,J(S)] - 2[S,E^0(\partial_x S)]\,, \qquad J(S) = \mathrm{tr}_2(m_{12}(0)S_2)\,.$$

Consider $N=2$ and elliptic case. Then $E^0=0$, and we come to the standard XYZ Landau-Lifshitz magnet, which integrability was proved by Sklyanin.

In the general (not elliptic and $N \neq 2$) case the obtained equation has the Zakharov-Shabat (or zero-curvature or Lax) representation

$$\partial_t U(z) - \partial_x V(z) + [U(z), V(z)] = 0\,,$$

where $U(z), V(z)$ is a pair of matrix-valued functions of the variables t, x depending also on the spectral parameter z:

$$U(z) = L(S, z) = \frac{1}{N}\,\mathrm{tr}_2\Big(r_{12}(z) S_2\Big)\,, \qquad V(z) = V_1(z) + V_2(z)\,,$$

$$V_1(z) = -c\partial_z L(S, z) + L(SE^0(S), z) + L(E^0(S)S, z)\,, \qquad V_2(z) = -cL(T, z)$$

and $T = -c^{-2}[S, \partial_x S]$ is a solution of equation $-\partial_x S = [S, T]$ for the case $S^2 = cS$.

Hamiltonian description. The Poisson structure is given by

$$\{S_{ij}(x), S_{kl}(y)\} = (S_{kj}(x)\delta_{il} - S_{il}(x)\delta_{kj})\delta(x-y) \quad \text{or} \quad \{S_1(x), S_2(y)\} = [P_{12}, S_1(x)]\delta(x-y)\,.$$

The equations of motion are reproduced as $\partial_t S(x) = \{H, S(x)\}$ the following Hamiltonian:

$$H^{\mathrm{LL}} = \oint dy \Big(\frac{c}{N}\;\mathrm{tr}\left(S\,J(S)\right) - \frac{1}{2c}\;\mathrm{tr}\left(\partial_y S\,\partial_y S\right) + \mathrm{tr}\left(\partial_y S\,E^0(S)\right)\Big)\,, \qquad S = S(y)\,.$$

To summarize, using construction of integrable tops as building blocks we obtained families of

- (classical) spin chains
- Sklyanin algebras
- Gaudin models
- KZ equations
- 1+1 Landau-Lifshitz models

Model of interacting tops.

It can be viewed as anisotropic version of the spin Calogero model: N interacting GL_M tops, i.e. $S_{ij} \to \mathcal{S}^{ij} \in \mathrm{Mat}_M$ and

$$\mathcal{H}^{\text{tops}} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} H^{\text{top}}(\mathcal{S}^{ii}) + \frac{1}{2} \sum_{i,j:\, i \neq j}^{N} \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j)\,.$$

$$H^{tops} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha \neq 0} S^{ii}_{\alpha} S^{ii}_{-\alpha} \wp(\omega_\alpha) - \frac{1}{2M} \sum_{i \neq j}^{N} \sum_{\alpha,\beta} \kappa^2_{\alpha,\beta} S^{jj}_{\beta} S^{ii}_{-\beta} \wp(\omega_\alpha + \frac{q_i - q_j}{M})\,.$$

$N = 1$ (a single block) case is the single Euler-Arnold top

$M = 1$ (each block is 1×1) case is the (spin) Calogero-Moser model

$$\mathcal{L}(z) = \left(\begin{array}{cccc} \mathcal{L}^{11}(z) & \mathcal{L}^{12}(z) & \ldots & \mathcal{L}^{1N}(z) \\ \mathcal{L}^{21}(z) & \mathcal{L}^{22}(z) & \ldots & \mathcal{L}^{2N}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^{N1}(z) & \mathcal{L}^{N2}(z) & \ldots & \mathcal{L}^{NN}(z) \end{array} \right) \Bigg\} \quad \text{in one column } N \text{ blocks of size } M \times M$$

Interacting tops through R-matrices solving AYBE.

$$L(z)=\sum_{i,j=1}^{N}E_{ij}\otimes L^{ij}(z)\,,\qquad L^{ij}(z)\in \mathrm{Mat}_M\qquad L(z)\in \mathrm{Mat}_{NM}\,,$$

$$L^{ij}(z)=\delta_{ij}\Big(p_i 1_M+\mathrm{tr}_2(\mathcal{S}_2^{ii}\,R_{12}^{z,(0)}P_{12})\Big)+(1-\delta_{ij})\,\mathrm{tr}_2(\mathcal{S}_2^{ij}\,R_{12}^{z}(q_{ij})P_{12})\,,\quad q_{ij}=q_i-q_j$$

and similarly for $M^{ij}(z)\in \mathrm{Mat}_M$

$$M^{ij}(z)=\delta_{ij}\,\mathrm{tr}_2(\mathcal{S}_2^{ii}\,R_{12}^{z,(1)}P_{12})+(1-\delta_{ij})\,\mathrm{tr}_2(\mathcal{S}_2^{ij}\,F_{12}^{z}(q_{ij})P_{12})\,,$$

where

$$F_{12}^{z}(u)=\partial_u R_{12}^{z}(u).$$

The Lax equations provide equations of motion, which comes from the Hamiltonian

$$\mathcal{H}^{\text{tops}}=\sum_{i=1}^{N}\frac{p_i^2}{2}+\frac{1}{2}\sum_{i=1}^{N}\mathrm{tr}_{12}\Big(m_{12}(0)\mathcal{S}_1^{ii}\mathcal{S}_2^{ii}\Big)+\sum_{i<j}^{M}\mathrm{tr}_{12}\Big(F_{12}^{0}(q_{ij})\mathcal{S}_1^{ii}\mathcal{S}_2^{jj}\Big)\,.$$

$$\{\mathcal{S}_{ab}^{ij},\mathcal{S}_{cd}^{kl}\}=\mathcal{S}_{cb}^{kj}\,\delta^{il}\,\delta_{ad}-\mathcal{S}_{ad}^{il}\,\delta^{kj}\,\delta_{bc}$$

or

$$\{\mathcal{S}_1^{ij},\mathcal{S}_2^{kl}\}=P_{12}\,\mathcal{S}_1^{kj}\,\delta^{il}-\mathcal{S}_1^{il}\,P_{12}\,\delta^{kj}\,.$$

Equations of motion:

$$\mathrm{tr}\left(\mathcal{S}^{ii}\right) = \mathrm{const}\,, \;\; \forall i$$

$$\dot{\mathcal{S}}^{ii} = [\mathcal{S}^{ii}, \mathrm{tr}_2(m_{12}(0)\mathcal{S}_2^{ii})] + \sum_{k:k\neq i}^{N} [\mathcal{S}^{ii}, \mathrm{tr}_2(F_{12}^0(q_{ik})\mathcal{S}_2^{kk})]\,,$$

$$\dot{p}_i = -\sum_{k:k\neq i}^{N} \mathrm{tr}_{12}\Big(\partial_{q_i} F_{12}^0(q_{ik})\mathcal{S}_1^{ii}\mathcal{S}_2^{kk}\Big)\,.$$

We will come back to this model at quantum level beginning with a different construction.

Relativistic interacting tops.

$$\mathcal{L}(z)=\sum_{i,j=1}^{M}E_{ij}\otimes\mathcal{L}^{ij}(z)\in\mathrm{Mat}(NM,\mathbb{C})\,,\quad \mathcal{L}^{ij}(z)\in\mathrm{Mat}(N,\mathbb{C})\,.$$

$$\mathcal{L}^{ij}(z)=\sum_{\alpha}T_\alpha\mathcal{S}^{ij}_\alpha\varphi_\alpha(z,\omega_\alpha+q_{ij}+\eta)\,,\quad q_{ij}=q_i-q_j\,,\quad \omega_\alpha=\frac{\alpha_1+\alpha_2\tau}{N}\,,$$

By introducing

$$J^{\eta,\,q_{ij}}(\mathcal{S}^{ij})=\sum_{\alpha}T_\alpha\mathcal{S}^{ij}_\alpha\Big(E_1(\omega_\alpha+q_{ij}+\eta)-E_1(\omega_\alpha+q_{ij})\Big)\,,\quad E_1(x)=\vartheta'(x)/\vartheta(x)$$

equations of motion take the form

$$\dot{\mathcal{S}}^{ij}=\mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj})-J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij}+\sum_{k:k\neq j}^{M}\mathcal{S}^{ik}J^{\eta,\,q_{kj}}(\mathcal{S}^{kj})-\sum_{k:k\neq i}^{M}J^{\eta,\,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\,.$$

$$\ddot{q}_i=\frac{1}{N}\,\mathrm{tr}\left(\dot{\mathcal{S}}^{ii}\right)=\frac{1}{N}\sum_{k:k\neq i}^{M}\mathrm{tr}\left(\mathcal{S}^{ik}J^{\eta,\,q_{ki}}(\mathcal{S}^{ki})-J^{\eta,\,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\right),$$

For $M=1$ case these are equations of motion introduced by Krichever and Zabrodin.
For $N=1$ one obtains relativistic top described by the classical Sklyanin algebra.

Part II: another application of AYBE – R-matrix valued Lax pairs.

Calogero-Moser model:

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i<j}^{N} \wp(q_i - q_j)\,, \qquad \wp(x) \to \frac{1}{\sin^2(x)} \to \frac{1}{x^2}$$

where ν – coupling constant, $\wp(q)$ – the Weierstrass $\wp$-function. Equations of motion are written in the Lax form with the following Lax matrix:

$$L(z) = \begin{pmatrix} p_1 & \nu\phi(z, q_1 - q_2) & \nu\phi(z, q_1 - q_3) & \dots & \nu\phi(z, q_1 - q_N) \\ \nu\phi(z, q_2 - q_1) & p_2 & \nu\phi(z, q_2 - q_3) & \dots & \nu\phi(z, q_2 - q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu\phi(z, q_N - q_1) & \nu\phi(z, q_N - q_2) & \nu\phi(z, q_N - q_3) & \dots & p_N \end{pmatrix}$$

where z – spectral parameter (I. Krichever 1980).

Let us replace the functions ϕ in the Lax matrix by R-matrices (with the Planck constant $\hbar \to z$)

$$\mathcal{L} = \sum_{a,b=1}^{N} E_{ab} \otimes \mathcal{L}_{ab}\,, \quad \mathcal{L}_{ab} = \delta_{ab} p_a\, 1 \otimes ...1 + \nu(1-\delta_{ab}) R^z_{ab}\,, \quad R^z_{ab} = R^z_{ab}(q_a - q_b)$$

or

$$\mathcal{L}(z) = \begin{pmatrix} p_1 1_n^{\otimes N} & \nu R^z_{12}(q_1-q_2) & \nu R^z_{13}(q_1-q_3) & \ldots & \nu R^z_{1N}(q_1-q_N) \\ \nu R^z_{21}(q_2-q_1) & p_2 1_n^{\otimes N} & \nu R^z_{23}(q_2-q_3) & \ldots & \nu R^z_{2N}(q_2-q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu R^z_{N1}(q_N-q_1) & \nu R^z_{N2}(q_N-q_2) & \nu R^z_{N3}(q_N-q_3) & \ldots & p_n 1_n^{\otimes N} \end{pmatrix}$$

and

$$\mathcal{M}_{ab}(z) = \nu \delta_{ab} d_a + \nu(1-\delta_{ab}) F^z_{ab}\,, \quad F^z_{ab} = \partial_{q_a} R^z_{ab}(q_a - q_b)\,,$$

where $d_a = -\sum_{c:\, c\neq a}^{N} F^0_{ac}\,, \quad F^0_{ac} = F^z_{ac}\,|_{z=0}\;.$

One can verify that the order of R-matrices is **incorrect**. It is **not in the agreement** with the identity $R^\hbar_{ab} F^\hbar_{bc} - F^\hbar_{ab} R^\hbar_{bc} = F^0_{bc} R^\hbar_{ac} - R^\hbar_{ac} F^0_{ab}$. **Instead**, we obtain $R^\hbar_{ac} F^0_{bc} - F^0_{ab} R^\hbar_{ac}$.

Let us add additional term to M-matrix:

$$\mathcal{M}(z) \to \mathcal{M}(z) + 1_N \otimes \mathcal{F}^{\,0}\,, \quad \mathcal{F}^{\,0} = \sum_{b,c:\, b>c}^{N} F^0_{bc} = \sum_{b,c:\, b>c}^{N} \partial_{q_b} r_{bc}(q_b - q_c) \in \mathrm{Mat}_M^{\otimes N}$$

In the scalar (Krichever's) case it does not effect the Lax equations. This term exactly changes the order of R and F^0

$$[R^z_{ac}, \mathcal{F}^{\,0}] + \sum_{b \neq a,c} R^z_{ab} F^z_{bc} - F^z_{ab} R^z_{bc} = \sum_{b \neq c} R^z_{ac} F^0_{bc} - \sum_{b \neq a} F^0_{ab} R^z_{ac}\,, \quad \forall\; a \neq c.$$

The the R-matrix identities work as it was expected.
The term $\mathcal{F}^{\,0}$ – is a matrix analogue of the Calogero-Moser potential

$$\mathcal{F}^{\,0} = \sum_{b,c:\, b>c}^{N} F^0_{bc}\,, \quad F^0_{bc} = \partial_{q_b} r_{bc}(q_b - q_c)\,.$$

In fact, this potential describes quantum version for interacting tops:

$$\mathcal{H}^{\text{tops}} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i=1}^{N} \mathrm{tr}_{12}\Big(m_{12}(0) \mathcal{S}_1^{ii} \mathcal{S}_2^{ii}\Big) + \sum_{i>j}^{M} \mathrm{tr}_{12}\Big(F^0_{12}(q_{ij}) \mathcal{S}_1^{ii} \mathcal{S}_2^{jj}\Big)\,.$$

We obtained the Lax equation in the form:

$$\dot{\mathcal{L}}(z) = \{H, \mathcal{L}(z)\} = [\mathcal{L}(z), \mathcal{M}(z) + 1_N \otimes \mathcal{F}^0]$$

Rewriting it as

$$\{H, \mathcal{L}(z)\} + [1_N \otimes \mathcal{F}^0, \mathcal{L}(z] = [\mathcal{L}(z), \mathcal{M}(z)]$$

we come to half-quantum Lax equation.

Many-body degrees of freedom (p_i and q_j) are classical, while the spin type variables are already quantum (in the fundamental representation).

Quantum long-range spin chains

Equilibrium position in the Calogero-Moser system: $p_j = 0$, and positions of particles q_j are fixed to be equidistant points on a circle $q_j = x_j = j/N$.

Then $\{H, \mathcal{L}(z)\} = 0$ and

$$\dot{\mathcal{L}}(z) = \{H, \mathcal{L}(z)\} = 0 = [\mathcal{L}(z), \mathcal{M}(z) + 1_N \otimes \mathcal{F}^0]$$

or

$$[1_N \otimes \mathcal{F}^0, \mathcal{L}(z)] = [\mathcal{L}(z), \mathcal{M}(z)]$$

where

$$[1_N \otimes \mathcal{F}^0, \mathcal{L}(z)] = \sum_{i,j=1}^{N} E_{ij} [\mathcal{F}^0, \mathcal{L}_{ij}]$$

<u>This is a quantum Lax equation with the quantum Hamiltonian $1 \otimes \mathcal{F}^0$!</u>

In some simple case $\mathcal{F}^0$ – is the Hamiltonian of the Haldane-Shastry long-range chain:

$$\mathcal{F}^0 = \sum_{i<j}^{N} \frac{P_{ij}}{\sin^2 \frac{\pi(i-j)}{N}} \,, \qquad P_{ij} = \frac{1}{2} \sum_{a=0}^{3} \overset{i}{\sigma}_a \overset{j}{\sigma}_a \quad \text{– spin exchange operator}\,.$$

Another simple example leads to the Inozemtsev chain $\mathcal{F}^0 = \sum_{i<j}^{N} P_{ij} \wp\Big(\frac{\pi(i-j)}{N}\Big)$.

For 8-vertex Baxter's R-matrix

$$R_{12}^{\hbar}(q) = \sum_{\alpha} \sigma_\alpha \otimes \sigma_\alpha \, \varphi_\alpha(q, \omega_\alpha + \hbar)$$

we get new anisotropic Hamiltonian extending the previous examples:

$$\mathcal{F}^0 \to \hat{H} = \sum_{i<j}^{N} \sum_{a=0}^{3} \overset{i}{\sigma}_a \overset{j}{\sigma}_a \; J_a(x_i - x_j) \,, \quad x_j = j/N$$

$$= \sum_{i<j} \Big(\overset{i}{\sigma}_0 \overset{j}{\sigma}_0 \; E_1'(x_{ij}) + \sum_{\alpha=1}^{3} \overset{i}{\sigma}_\alpha \overset{j}{\sigma}_\alpha \; \varphi_\alpha(x_{ij})(E_1(x_{ij} + \omega_\alpha) - E_1(x_{ij}) - E_1(\omega_\alpha)) \Big) \,,$$

Possible solutions of the associative Yang-Baxter equations provide a wide class of new quantum integrable long-range spin chains.

Is it integrable?

How to find higher Hamiltonians?

All the flows are described by the Lax equations

$$\partial_{t_k} L(z) \equiv \{H_k, L(z)\} = [L(z), M^{(k)}(z)]\,,$$

Third flow in Calogero-Moser model

The third Hamiltonian

$$H_3 = \sum_{i=1}^{N} \frac{p_i^3}{3} - \nu^2 \sum_{i\neq j}^{N} p_i\, \wp(q_i - q_j)$$

provides equations of motion

$$\begin{cases} \partial_{t_3} q_i = p_i^2 - \nu^2 \sum\limits_{k\neq i} \wp(q_i - q_k)\,, \\ \partial_{t_3} p_i = \nu^2 \sum\limits_{k\neq i} (p_i + p_k) \wp'(q_i - q_k)\,. \end{cases}$$

M-matrix is of the form:

$$M_{ij}^{(3)}(z) = -\delta_{ij}\,\nu \sum_{k\neq i}(p_i + p_k)f(0,q_{ik}) +$$

$$+(1-\delta_{ij})\Big(\nu(p_i+p_j)f(z,q_{ij}) + \nu^2 \sum_{k\neq i,j}(\phi(z,q_{ik})f(z,q_{kj}) - \phi(z,q_{ij})f(0,q_{kj}))\Big)$$

R-matrix-valued generalization of the M-matrix for the third flow:

$$\mathcal{M}_{ij}^{(3)}(z) = -\delta_{ij}\,\nu \sum_{k\neq i}(p_i + p_k)F_{ik}^0(q_{ik}) +$$

$$+(1-\delta_{ij})\Big(\nu(p_i+p_j)F_{ij}^z(q_{ij}) + \nu^2 \sum_{k\neq i,j}(R_{ik}^z(q_{ik})F_{kj}^z(q_{kj}) - R_{ij}^z(q_{ij})F_{kj}^0(q_{kj}))\Big)+$$

$$+\delta_{ij}\Big(\nu^2 {\sum_{b,c}}' [F_{bc}^0(q_{bc}), r_{ic}(q_{ic})] + \nu {\sum_{b,c}}' p_b F_{bc}^0(q_{bc}) - \frac{\nu^2}{3} {\sum_{a,b,c}}' [F_{ab}^0(q_{ab}), r_{cb}(q_{cb})]\Big)$$

The analogue of $\mathcal{F}^0$-term gives

$$\mathcal{H}_3^{\text{chain}} = \sum_{a,b,c}{}' [F^0_{ab}(x_{ab}), r_{cb}(x_{cb})] \,, \quad x_j = j/N$$

It was originally verified numerically that

$$[\mathcal{H}_2^{\text{chain}}, \mathcal{H}_3^{\text{chain}}] = 0 \,.$$

Now we know several ways to derive the Hamiltonians and prove commutativity. One of them was presented yesterday by M. Matushko through Dunkl operators.
Another way – explicit construction of q-deformed commuting set of operators.

We started from the classical many-body system and now are discussing a quantum spin chain. How it happened?

$$\{H, \mathcal{L}\} + [\nu \mathcal{F}^0, \mathcal{L}(z)] = [\mathcal{L}(z), \mathcal{M}(z)]\,,$$

where $\mathcal{M}$ does not include the $\mathcal{F}^0$ term. In this respect the R-matrix-valued Lax pair is "half-quantum": the spin variables are quantized in the fundamental representation, while the positions and momenta of particles remain classical. The $\mathcal{F}^0$ term in this treatment is the (anisotropic) spin exchange operator.
In the equilibrium position $\{H, \mathcal{L}\} = 0$ we get quantum Lax equation.
The quantities $\mathrm{tr}(\mathcal{L}^k)$ are the classical (Calogero-Moser) Hamiltonians only.

The $\mathcal{F}^0$-term describes the quantum model of interacting tops (spins).

N interacting gl_M tops

$$\mathcal{H}^{\text{tops}} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} H^{\text{top}}(\mathcal{S}^{ii}) + \frac{1}{2} \sum_{i,j:\, i\neq j}^{N} \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j)\,.$$

$$H^{tops} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha\neq 0} S_{\alpha}^{ii} S_{-\alpha}^{ii} \wp(\omega_\alpha) - \frac{1}{2M} \sum_{i\neq j}^{N} \sum_{\alpha,\beta} \kappa_{\alpha,\beta}^2 S_{\beta}^{jj} S_{-\beta}^{ii} \wp(\omega_\alpha + \frac{q_i - q_j}{M})\,.$$

$N = 1$ case – single Euler-Arnold top
$M = 1$ case – the Calogero-Moser model

Under quantization the potential $\frac{1}{2} \sum_{i,j:\, i\neq j}^{N} \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j)$ turns into $\mathcal{F}^0$, which we added to $\mathcal{M}(z)$ to have appropriate order of R-matrices:

$$\hat{\mathcal{H}}^{\text{tops}} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2} + \sum_{i=1}^{N} \hat{H}^{\text{top}}(\hat{\mathcal{S}}^{ii}) + \mathcal{F}^0\,.$$

One more origin of $\mathcal{F}^0$-term – elliptic Knizhnik-Zamolodchikov equations:

Consider the gl_N elliptic KZ equations for N punctures on elliptic curve with moduli τ:

$$\nabla_i \psi = 0\,, \quad \nabla_i = \partial_i + \nu \sum_{j:j\neq i} r_{ij}(z_i - z_j)\,,$$

for $i = 1, ..., N$ and

$$\nabla_\tau \psi = 0\,, \quad \nabla_\tau = 2\pi\imath \partial_\tau + \frac{\nu}{2} \sum_{j \neq k} m_{jk}(z_j - z_k)\,,$$

where r_{ij} and m_{ij} are the coefficients of the expansion of R-matrix. Commutativity $[\nabla_i, \nabla_j] = 0$ follows from the classical Yang-Baxter equation, while $[\nabla_i, \nabla_\tau] = 0$ follows from AYBE and the heat equation $(2\pi\imath\partial_\tau R^\hbar_{12}(z) = \partial_z \partial_\hbar R^\hbar_{12}(z))$:

$$2\pi\imath \partial_\tau r_{ab} = \partial_{z_a} m_{ab}\,, \qquad [r_{ab}, m_{bc} + m_{ac}] + [r_{ac}, m_{ab} + m_{bc}] = 0\,.$$

Then ψ satisfies also non-stationary Schrödinger equation

$$\left(2\pi\imath N \nu \partial_\tau + \frac{1}{2}\Delta\right)\psi = \left(-\nu \mathcal{F}^0 - \frac{1}{2} N \nu^2 \sum_j m_{jj} + \nu^2 N^2 \,\mathrm{Id} \sum_{i<j} \wp(z_i - z_j)\right)\psi\,,$$

where $\Delta = \sum_i \partial_i^2$ and $m_{jj} = m_{jj}(0)$ are scalar operators depending on τ.

Relativistic models and the Uglov's type q-deformation of long-range spin chains.

Introduce the operators $\mathcal{D}_k$ (matrix generalization of the scalar operators acting in $\mathrm{End}(\mathcal{H})$):

$$\mathcal{D}_k = \sum_{1\leq i_1<\ldots<i_k\leq N} \left(\prod_{\substack{j\,=\,1\\ j\neq i_1\ldots i_{k-1}}}^{N} \phi(z_j - z_{i_1},\hbar)\, \phi(z_j - z_{i_2},\hbar)\, \cdots\, \phi(z_j - z_{i_k},\hbar) \right) \times$$

$$\times \left(\overleftarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{j_1 i_1} \overleftarrow{\prod_{\substack{j_2\,=\,1\\ j_2\neq i_1}}^{i_2-1}} \bar{R}_{j_2 i_2} \ldots \overleftarrow{\prod_{\substack{j_k\,=\,1\\ j_k\neq i_1\ldots i_{k-1}}}^{i_k-1}} \bar{R}_{j_k i_k} \right) \times$$

$$\times e^{-\eta\partial_{z_{i_1}}} \cdots e^{-\eta\partial_{z_{i_k}}} \times \left(\overrightarrow{\prod_{\substack{j_k=1\\ j_k\neq i_1\ldots i_{k-1}}}^{i_k-1}} \bar{R}_{i_k j_k} \overrightarrow{\prod_{\substack{j_{k-1}=1\\ j_{k-1}\neq i_1\ldots i_{k-2}}}^{i_{k-1}-1}} \bar{R}_{i_{k-1} j_{k-1}} \ldots \overrightarrow{\prod_{j_1=1}^{i_1-1}} \bar{R}_{i_1 j_1} \right),$$

where $k = 1, \ldots, N$ and $\bar{R}_{ij} = \bar{R}^{\hbar}_{ij}(z_i - z_j)$. For a wide class of R-matrices these operators commute

$$[\mathcal{D}_i, \mathcal{D}_j] = 0\,.$$

In the scalar case ($\bar{R}_{ij} = 1$) these spin operators coincide with the Macdonald-Ruijsenaars operators:

$$D_k = \sum_{|I|=k} \prod_{\substack{i \in I \\ j \notin I}} \phi(\hbar, z_j - z_i) \prod_{i \in I} e^{-\eta \partial_{z_i}}, \qquad k = 1, \ldots, N. \tag{1}$$

At classical level these are Hamiltonians of relativistic interacting tops on GL_{NM} Lie group. By introducing

$$J^{\eta,\, q_{ij}}(\mathcal{S}^{ij}) = \sum_{\alpha} T_\alpha \mathcal{S}^{ij}_\alpha \Big(E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij}) \Big)\,, \quad E_1(x) = \vartheta'(x)/\vartheta(x)$$

equations of motion take the form

$$\dot{\mathcal{S}}^{ij} = \mathcal{S}^{ij} J^{\eta}(\mathcal{S}^{jj}) - J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij} + \sum_{k:k\neq j}^{N} \mathcal{S}^{ik} J^{\eta,\, q_{kj}}(\mathcal{S}^{kj}) - \sum_{k:k\neq i}^{N} J^{\eta,\, q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\,.$$

$$\ddot{q}_i = \frac{1}{N} \operatorname{tr}\left(\dot{\mathcal{S}}^{ii}\right) = \frac{1}{N} \sum_{k:k\neq i}^{N} \operatorname{tr}\left(\mathcal{S}^{ik} J^{\eta,\, q_{ki}}(\mathcal{S}^{ki}) - J^{\eta,\, q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\right),$$

For $M = 1$ case these are equations of motion for the spin Ruijsenaars-Schneider model introduced by Krichever and Zabrodin.
For $N = 1$ one obtains relativistic top described by the classical Sklyanin algebra.

Return back to the associative Yang-Baxter equation, which can be viewed as quadratic algebra

$$r_{ij}r_{jk} = r_{ik}r_{ij} + r_{jk}r_{ik} \qquad \text{for distinct } i, j, k\,.$$

It was extended by An. Kirillov to B-type associative Yang-Baxter algebra, which includes relations

$$r_{ij}y_j = y_i r_{ij} + \tilde{r}_{ij}y_i + y_j\tilde{r}_{ij}\,.$$

with additional generators y_i.

In our recent paper with M. Matushko and A. Mostovskii we showed that this algebra has representation, where the generators y_i become the **boundary K-matrices** solving the reflection equation

$$R^{-}_{12}(x_1,x_2)K^{\hbar}_{1}(x_1)R^{+}_{12}(x_1,x_2)K^{\hbar}_{2}(x_2) = K^{\hbar}_{2}(x_2)R^{+}_{12}(x_1,x_2)K^{\hbar}_{1}(x_1)R^{-}_{12}(x_1,x_2)\,,$$

where $R^{-}_{12}(x_1,x_2) = R^{\hbar}_{12}(x_1 - x_2)$, $R^{+}_{12}(x_1,x_2) = R^{\hbar}_{12}(x_1 + x_2)$. The K-matrices play the role of the boundary conditions in quantum integrable systems.

Namely,

$$R_{ij}^{w+z}(q_i - q_j)\tilde{K}_j^w(q_j) = \tilde{K}_i^w(q_i)R_{ij}^{z-w}(q_i - q_j) + \tilde{R}_{ij}^{w-z}(q_i + q_j)\tilde{K}_i^z(q_i) + \tilde{K}_j^{-z}(q_j)\tilde{R}_{ij}^{w+z}(q_i + q_j)\,.$$

For example, for the Baxter's 8-vertex R-matrix we have

$$\tilde{K}^{\hbar}(z) = \sum_{k=0}^{3} \nu_k e^{2\pi\imath(z+\hbar+\omega_k)\partial_\tau \omega_k} \phi(z+\omega_k, \hbar+\omega_k)\sigma_{4-k}\,,$$

This allows to apply construction of R-matrix valued pairs to BC_n type Calogero-Inozemtsev system:

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 - g^2 \sum_{i<j}^{n} \Big(\wp(q_i - q_j) + \wp(q_i + q_j)\Big) - \frac{1}{2}\sum_{a=0}^{3}\sum_{k=1}^{n} \nu_a^2 \wp(q_k + \omega_a)\,,$$

where ω_γ are half-periods, and the five arbitrary constants are $g, \nu_0, \nu_1, \nu_2, \nu_3 \in \mathbb{C}$.

The Takasaki's $2n \times 2n$ Lax pair has a natural block-matrix structure:

$$L(z) = \begin{pmatrix} L^{11}(z) & L^{12}(z) \\ L^{21}(z) & L^{22}(z) \end{pmatrix}, \qquad \begin{array}{l} L^{11}_{ij}(z) = \delta_{ij} p_i + g(1-\delta_{ij})\phi(z, q_{ij})\,, \\ L^{12}_{ij}(z) = \delta_{ij} v(z, q_i) + g(1-\delta_{ij})\phi(z, q^{+}_{ij})\,, \\ L^{21}_{ij}(z) = -\delta_{ij} v(-z, q_i) - g(1-\delta_{ij})\phi(-z, q^{+}_{ij})\,, \\ L^{22}_{ij}(z) = -\delta_{ij} p_i - g(1-\delta_{ij})\phi(-z, q_{ij}) \end{array}$$

The function $v(z, q_i)$

$$v(z,u) \equiv v(z,u|\nu) = \sum_{a=0}^{3} \nu_a \exp(4\pi\imath z \partial_\tau \omega_a)\phi(2z, u+\omega_a)$$

is generalized to K-matrix, while ϕ-function are again replaced with R-matrices.
In this way one finds BC_n analogue for $\mathcal{F}^0$-term is

$$\mathcal{H} = g\sum_{k<l}^{n}\Big(F^0_{kl}(q_k - q_l) + \tilde{F}^0_{kl}(q_k+q_l)\Big) + \frac{1}{2}\sum_{k=1}^{n}\tilde{Y}^0_k(q_k)\,, \qquad \tilde{Y}^0(x) = \partial_x K^{\hbar}(x)\Big|_{\hbar=0}$$

which now describes a new family of integrable long-range spin chains with boundaries.

Thank you!